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Article

Unified Connection and Curvature Formulas on Singly, Doubly, and Sequential Warped Product Manifolds

Salah Agha[1,*], **Ahmed K. Elsherif** [1], **Alhaytham Aref** [1], **and Sameh Shenawy** [2]

[1] *Mathematics Department, Military Technical College, Cairo, Egypt*
[2] *Basic Science Department, Modern Academy for Engineering and Technology, Maadi 4411602, Egypt*

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ABSTRACT

This paper presents a unified differential-geometric treatment of warped product manifolds designed for explicit connection and curvature analysis. After establishing the manifold, tangent, metric, and covariant-derivative framework, the study formalizes Lie-derivative criteria for Killing, 2-Killing, and conformal vector fields and organizes the principal curvature invariants, namely Riemann, sectional, Ricci, and scalar curvature. A complete computation on the round unit sphere validates the framework by recovering the expected Levi-Civita structure, constant sectional curvature 1, Ricci tensor equal to the metric, and scalar curvature 2. Building on this foundation, the manuscript derives systematic Levi-Civita identities, Lie-derivative decompositions, and curvature formulas for singly warped, doubly warped, and sequential warped products. The resulting expressions isolate the roles of base and fibre geometry, Hessian terms, and warping-function gradients in pure and mixed curvature components. These results provide a consistent reference for layered warped geometries and support applications in geometric analysis, Einstein-type equations, and relativity-motivated geometric models

1. Introduction

Differential geometry provides the intrinsic framework for analysing smooth spaces through coordinate-independent objects such as manifolds, tangent bundles, Riemannian metrics, and

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* **Corresponding author**
E-mail address: salahagha@mtc.edu.eg

affine connections. This framework is fundamental for defining geodesics, covariant differentiation, and curvature in a mathematically rigorous way [1-3]. Beyond pure mathematics, these structures are central to physical modelling, especially in relativity, where geometric invariants encode physical laws and the metric determines causal and dynamical behaviour [3,4].

Within this setting, warped product manifolds have become a major construction tool because they allow controlled coupling of two or more geometric factors through warping functions. The classical warped product formalism introduced by Bishop and O'Neill established how curvature and connection terms split between base and fibre components [5]. Later developments extended this framework to twisted and doubly warped models, enabling richer anisotropic geometries and broader pseudo-Riemannian applications [6-8]. Sequential warped products further generalize this idea by applying warping in multiple stages, producing layered curvature interactions that are useful in geometric analysis and Einstein-type modelling [9,10].

A related direction concerns geometric vector fields and metric deformations. In particular, Killing and 2-Killing structures provide symmetry constraints that connect Lie derivatives of the metric with conservation-type properties and geometric rigidity [11]. At the same time, Ricci and scalar curvature quantities remain central in global geometric analysis, comparison theorems, and curvature evolution problems [12,13]. These themes motivate a unified treatment in which local differential operators, symmetry conditions, and curvature decompositions are presented within one consistent notation.

Motivated by these developments, this paper develops an integrated differential-geometric presentation that begins with manifold and connection preliminaries and then derives explicit formulas for Levi-Civita connection terms, Lie derivatives, and principal curvature tensors on singly warped, doubly warped, and sequential warped manifolds. The objective is to provide a coherent reference that clarifies how warping functions govern mixed and pure curvature components, while keeping the framework directly usable for geometric analysis and relativity-oriented applications [5-9].

2. Differential Manifold

Differential manifold theory extends calculus from Euclidean spaces to spaces that are locally Euclidean but may have nontrivial global geometry. Its central aim is to provide a rigorous local-to-global framework for smooth maps, curves, vector fields, and tensor fields [1,14,15,16,17].

Let $n \in \mathbb{N}$ and

$$\mathbb{R}^n = \{(p^1, \dots, p^n): p^i \in \mathbb{R}\}$$

For each $i = 1, \dots, n$, define the canonical coordinate projection $u^i: \mathbb{R}^n \to \mathbb{R}$ by

$$u^i(p) = p^i, p = (p^1, \dots, p^n)$$

If B is a topological space and $\varphi: U \to \varphi(U) \subset \mathbb{R}^n$, we write

$$\varphi = (x^1, \dots, x^n), x^i = u^i \circ \varphi$$

and call $(x^1, \dots, x^n)$ the local coordinate functions on U.A coordinate chart on B is a pair (U, φ) where $U \subset B$ is open and $\varphi: U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$; the set U is a coordinate neighbourhood (or coordinate patch). If (U, φ) and (V, ψ) are two n-dimensional charts with $U \cap V \neq \emptyset$, their transition maps

$$\psi \circ \varphi^{-1}: \varphi(U \cap V) \to \psi(U \cap V), \varphi \circ \psi^{-1}: \psi(U \cap V) \to \varphi(U \cap V)$$

must be smooth. In this case, the charts are said to be smoothly compatible on the overlap. An atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ on B is a family of charts with $\cup_\alpha U_\alpha = B$. The atlas is smooth when every transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is C^∞ on its domain. A smooth atlas is maximal if it contains every chart smoothly compatible with it. The overlap structure and transition maps are illustrated in Figure 1.

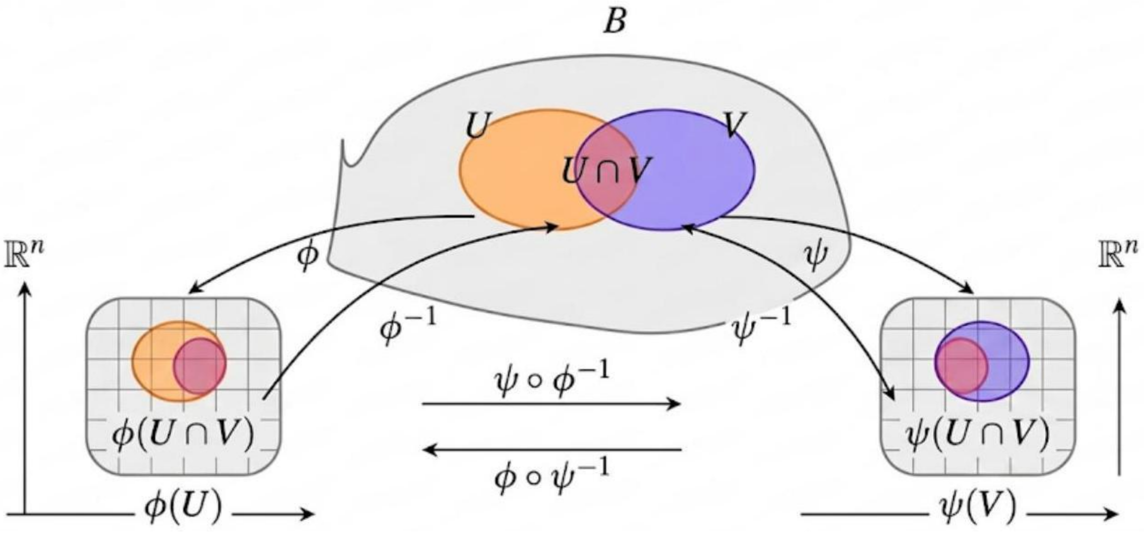

Figure 1. Two overlapping coordinate neighbourhoods and transition maps.

Throughout this paper, unless explicitly stated otherwise, B denotes an n-dimensional manifold.

An n-dimensional differential manifold is a pair ($B, \mathcal{A}$) such that:

1. B is Hausdorff;
2. B is second countable;
3. $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas on B with $\bigcup_\alpha U_\alpha = B$, each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ a homeomorphism onto an open subset of $\mathbb{R}^n$, and all transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ smooth on overlaps.

Since B is locally homeomorphic to $\mathbb{R}^n$, it inherits local properties of Euclidean space, including local compactness and local connectedness.

3. Tangent Space and Tangent Bundles

Tangent vectors provide the infinitesimal framework that extends directional differentiation from $\mathbb{R}^n$ to smooth manifolds. In Euclidean space, derivatives encode both direction and rate of change; at a point $p \in B$, elements of T_pB play the same role independently of coordinates. They act on smooth functions as derivations and represent velocities of smooth curves through p. This local linear structure is fundamental for vector fields, first-order differential operators, and geometric objects such as connections and curvature.

Let B be a smooth manifold. We denote by $\mathcal{F}(B) = C^\infty(B)$ the set of all smooth real-valued functions on B. A function $f : B \to \mathbb{R}$ is smooth when, for every chart (U, φ) on B, the coordinate expression

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$$

is C^∞ in the Euclidean sense. If $f, g \in \mathcal{F}(B)$, then $f + g \in \mathcal{F}(B)$ and $fg \in \mathcal{F}(B)$. For regular surfaces in $\mathbb{R}^3$, a tangent vector at p is the velocity of a smooth curve passing through p. The same interpretation extends to smooth manifolds: if $c: (-\varepsilon, \varepsilon) \to B$ with $c(0) = p$, then c determines a directional action on smooth functions through the derivative of $f \circ c$ at 0 . This geometric viewpoint motivates the intrinsic algebraic formulation below, which is independent of any embedding.

A tangent vector at a point $p \in B$ is a map $X_p: \mathcal{F}(B) \to \mathbb{R}$ such that, for all $f, g \in \mathcal{F}(B)$ and $r, s \in \mathbb{R}$,

$$X_p(rf + sg) = rX_p(f) + sX_p(g)$$

and

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$$

For $p \in B$, the tangent space at p is the set

$$T_pB = \{X_p: \mathcal{F}(B) \to \mathbb{R} \mid X_p \text{ satisfies linearity and the Leibniz rule}\}$$

With the operations

$$(X_p + Y_p)(f) = X_p(f) + Y_p(f), (\lambda X_p)(f) = \lambda X_p(f)$$

T_pB is a real vector space. Its geometric interpretation at a point is shown in Figure 2.

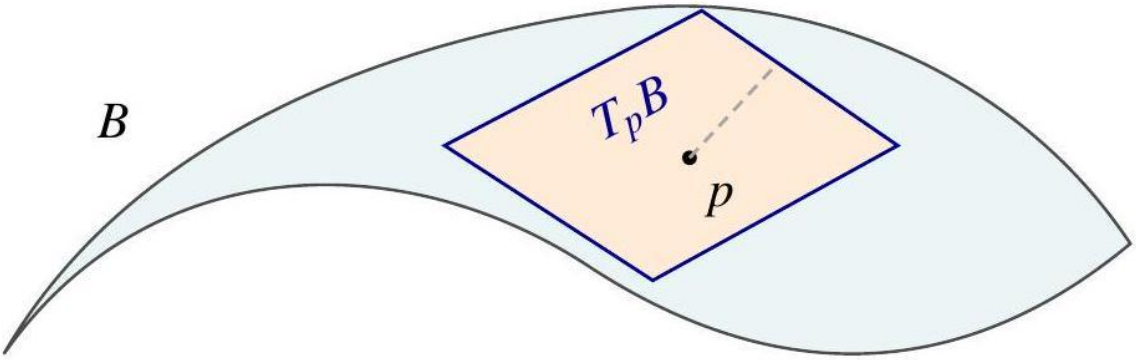

Figure 2. Tangent space T_pB at a point $p \in B$.

The tangent bundle of B is the union of all tangent spaces; mathematically, it is given by the disjoint union

$$TB = \bigsqcup_{p \in B} T_pB,$$

equipped with the canonical projection $\pi: TB \to B$, defined by $\pi(v) = p$ for $v \in T_pB$. Thus each fiber is

$$\pi^{-1}(p) = T_pB,$$

So TB is a rank- n real vector bundle over B. It carries a natural smooth manifold structure of dimension $2n$. For a local chart $(U, x^1, \dots, x^n)$ on B, the induced coordinates on $\pi^{-1}(U)$ are

$$v \upharpoonright\longrightarrow \left(x^1(p), \dots, x^n(p), v(x^1), \dots, v(x^n)\right) \in \mathbb{R}^{2n}, v \in T_pB.$$

Hence, locally,

$$\pi^{-1}(U) \cong U \times \mathbb{R}^n.$$

On overlaps, the induced coordinate changes are smooth and linear in the fiber variables, with fiber transformation given by the Jacobian of the base coordinate change.

After defining the tangent bundle, we turn to vector fields, which are fundamental in differential geometry and widely used in mechanics, fluid dynamics, and electromagnetism. In particular, electric fields are modelled by vector fields (for electrostatics, typically as gradient fields of scalar potentials). Common classes studied in geometry and applications include gradient, divergence-free, Killing, and conformal vector fields. Mathematically, a vector field on B is a section of the tangent bundle.

A smooth vector field on B is a smooth cross-section of the tangent bundle projection $\pi: TB \to B$, that is, a smooth map

$$X: B \to TB, \pi \circ X = \mathrm{id}_B$$

The set of all smooth vector fields on B is denoted by $\mathfrak{X}(B)$.In a local chart $(U, x^1, \dots, x^n)$, write $\partial_i := \frac{\partial}{\partial x^i}$. For any smooth function f, one has $\partial_i(f) = \frac{\partial f}{\partial x^i}$, and every $X \in \mathfrak{X}(U)$ has the unique form

$$X = \sum_{i=1}^{n} X^i \partial_i, X^i = X\left(x^i\right)$$

A derivation on $C^\infty(B)$ is a map $D: C^\infty(B) \to C^\infty(B)$ such that

$$D(af + bg) = aD(f) + bD(g), D(fg) = D(f)g + fD(g),$$

for all $a, b \in \mathbb{R}$ and $f, g \in C^\infty(B)$. Each smooth vector field defines a derivation by $f \mapsto X(f)$, and conversely every derivation on $C^\infty(B)$ is induced by a unique smooth vector field.

For $X, Y \in \mathfrak{X}(B)$, the Lie bracket is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)), f \in C^\infty(B).$$

This again defines a smooth vector field. In local coordinates,

$$[X, Y] = \left(X^i \frac{\partial Y^k}{\partial x^i} - Y^i \frac{\partial X^k}{\partial x^i}\right) \partial_k$$

It is bilinear, skew-symmetric, and satisfies the Jacobi identity. For example, on $\mathbb{R}^2$ with

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, Y = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

one obtains $[X, Y] = 0$.

These constructions provide the local linear model used throughout differential geometry [7,9].

To differentiate one vector field along another, one needs additional structure on TB, namely a linear connection. This notion is fundamental for parallel transport, geodesics, and curvature, and it is therefore central in geometry and in physical models based on field equations.

A linear connection on B is a map

$$\nabla : \mathfrak{X}(B) \times \mathfrak{X}(B) \to \mathfrak{X}(B), (X, Y) \mapsto \nabla_X Y,$$

such that for all $X, Y, Z \in \mathfrak{X}(B)$ and $f, g \in C^\infty(B)$:

$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z \tag{1}$$

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \tag{2}$$

$$\nabla_X (fY) = X(f)Y + f\nabla_X Y \tag{3}$$

In local coordinates $(x^1, \ldots, x^n)$, it is determined by

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$$

where $\Gamma^k{}_{ij}$ are the connection coefficients.The coefficients $\Gamma^k{}_{ij}$ are called the Christoffel symbols of the connection ∇ in the coordinate system $(x^1, \ldots, x^n)$. They encode the local action of ∇ on the coordinate frame $\{\partial_i\}$, and are smooth functions on the coordinate domain. Their values depend on the chosen coordinates and on the chosen connection.

As a basic example, on $\mathbb{R}^n$ with standard coordinates, the canonical flat connection is given by $\Gamma^k{}_{ij} = 0$, hence

$$\nabla_X Y = \left(X^i \frac{\partial Y^k}{\partial x^i} \right) \partial_k$$

For this connection, parallel transport is path-independent, and geodesics are straight lines.

4. The Metric of the Manifold

A differentiable manifold carries a smooth structure, but this structure alone does not define intrinsic notions of length, angle, or distance. To measure these quantities on a curved space, one equips each tangent space with a bilinear pairing that varies smoothly from point to point. This additional structure is the metric, and it is the basis for geodesics, curvature, and volume.

Let V be a real vector space. A scalar product on V is a symmetric non-degenerate bilinear form

$$b : V \times V \to \mathbb{R}$$

With respect to a basis $\{e_i\}$, it can be written as $b(u, v) = b_{ij} u^i v^j$, where (b_{ij}) is symmetric and invertible. If $b(v, v) > 0$ for every $v \neq 0$, then b is positive-definite and recovers the Euclidean inner-product model; if the signature is indefinite, it provides the algebraic model used in pseudo-Riemannian geometry.

Accordingly, a metric tensor on B is a smooth assignment $p \mapsto g_p$, where each

$$g_p : T_p B \times T_p B \to \mathbb{R}$$

is an inner product. For each $p \in B, X_p, Y_p, Z_p \in T_p B$, and $r \in \mathbb{R}$, the following properties hold:

1. (Symmetry) $g_p(X_p, Y_p) = g_p(Y_p, X_p)$.
2. (Bilinearity) $g_p(X_p + Y_p, Z_p) = g_p(X_p, Z_p) + g_p(Y_p, Z_p)$ and $g_p(rX_p, Y_p) = r g_p(X_p, Y_p)$.
3. (Positive-definiteness) $g_p(X_p, X_p) \geq 0$, with equality if and only if $X_p = 0$.
4. (Smoothness) If X, Y are C^∞ vector fields on an open set $A \subset B$, then

$$p \mapsto g_p(X_p, Y_p)$$

Is a C^∞ function on A.

When each g_p is positive-definite, (B, g) is a Riemannian manifold. When g is non-degenerate with constant indefinite signature $(k, n-k)$, (B, g) is pseudo-Riemannian; Lorentzian geometry is the important case $k = 1$ used in spacetime models [3,18,119].

In local coordinates $(x^1, \dots, x^n)$:

$$g = g_{ij} dx^i \otimes dx^j, ds^2 = g_{ij} dx^i dx^j \tag{4}$$

If ∇ is the Levi-Civita connection of g, then its Christoffel symbols are determined by the metric coefficients:

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \right)$$

A connection ∇ on TB is called metric (or metric-compatible) if it preserves the metric under covariant differentiation, namely

$$(\nabla_X g)(Y, Z) = 0, X, Y, Z \in \mathfrak{X}(B),$$

equivalently,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

For a Riemannian manifold, the Levi-Civita connection is the unique torsion-free metric connection.

The next section introduces the covariant derivative and its action on vector fields.

5. Connections and Covariant Derivative

On a smooth manifold, tangent spaces at different points are distinct, so the ordinary derivative of one vector field along another is not intrinsically defined. This motivates the covariant derivative, namely a connection

$$\nabla : \mathfrak{X}(B) \times \mathfrak{X}(B) \to \mathfrak{X}(B), (X, Y) \mapsto \nabla_X Y,$$

which differentiates Y in the direction of X. A linear connection satisfies, for $X, Y, Z \in \mathfrak{X}(B)$ and $f, g \in C^\infty(B)$,

$$\begin{aligned} \nabla_{fX+gY} Z &= f \nabla_X Z + g \nabla_Y Z \\ \nabla_X (Y + Z) &= \nabla_X Y + \nabla_X Z \\ \nabla_X (fY) &= X(f) Y + f \nabla_X Y \end{aligned} \tag{5}$$

Its torsion is

$$T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y].$$

The connection is called symmetric if $T=0$, equivalently $\nabla_X Y-\nabla_Y X=[X,Y]$. Thus, the laws of a linear connection are given by (5). The Levi-Civita connection is the unique linear connection satisfying, in addition, the two laws

1. torsion-free: $\nabla_X Y-\nabla_Y X=[X,Y]$;
2. metric-compatible: $X(g(Y,Z))=g(\nabla_X Y,Z)+g(Y,\nabla_X Z)$.
   for all $X,Y,Z\in\mathfrak{X}(B)$[1,9].

6. Curves and Vector Fields on the Manifolds

The study of curves on manifolds is fundamental because curves provide the most direct way to examine local and global geometry. Through curves one defines velocity, acceleration, geodesics, length, and parallel transport, and one obtains a concrete link between intrinsic geometric structure and dynamical models on B[1,9].

Let $I\subset\mathbb{R}$ be an open interval. A (smooth) curve on B is a smooth map $\gamma: I\rightarrow B$. For each $t\in I$, its velocity is a tangent vector $\dot{\gamma}(t)\in T_{\gamma(t)}B$, characterized by

$$\dot{\gamma}(t)(f)=\frac{d}{dt}(f\circ\gamma)(t), f\in C^{\infty}(B)$$

Thus a curve assigns to each parameter value a point of B together with a canonical tangent direction along its image.

Among smooth curves, geodesics play a central role. They provide the intrinsic notion of straight motion on a manifold and are fundamental in geometric analysis, mechanics, and relativity. In Riemannian geometry they model locally length-minimizing trajectories, while in applications they are used in shortest-path problems, interpolation on manifolds, and geometric modelling [20,21].

Given a connection ∇ on B, a smooth curve $\gamma: I\rightarrow B$ is called a geodesic when its velocity field is parallel along the curve, that is,

$$\nabla_{\dot{\gamma}}\dot{\gamma}=0.$$

In local coordinates $x^k(t)$, this condition is equivalent to the geodesic system

$$\frac{d^2x^k}{dt^2}+\Gamma_{ij}^k(x(t))\frac{dx^i}{dt}\frac{dx^j}{dt}=0, k=1,\dots,n$$

On the unit sphere

$$S^2=\{(x,y,z)\in\mathbb{R}^3: x^2+y^2+z^2=1\}$$

a smooth curve is, for example, the latitude

$$\gamma_h(t)=\left(\sqrt{1-h^2}\cos\, t,\sqrt{1-h^2}\sin\, t,h\right), |h|<1.$$

This is a regular curve on S^2; it is geodesic only in the case $h = 0$ (the equator). A geodesic curve on S^2 is any great circle. In explicit vector form, one may write

$$\sigma(t) = (\cos t \quad \sin t \quad 0)$$

which lies on S^2 for all t. More generally, every great circle has the form

$$\sigma(t) = \cos tu + \sin tv,$$

where $u, v \in \mathbb{R}^3$ are orthonormal vectors. Hence $\sigma(t) \in S^2$ for all t, and its image is $S^2 \cap \operatorname{span}\{u, v\}$, a great circle [20,21].

The study of vector fields on manifolds is equally fundamental, since a vector field assigns an infinitesimal direction at each point and therefore models flows, symmetries, transport phenomena, and dynamical systems on B. In both geometric analysis and applications, vector fields connect local differential structure with global behaviour through integral curves and induced transformations [20,21].

For a vector field X, the Lie derivative of the metric can be written in covariant-derivative form as

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X), Y, Z \in \mathfrak{X}(B),$$

or equivalently in coordinates,

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i.$$

For a vector field X, the Lie derivative $\mathcal{L}_X$ describes the infinitesimal change of tensor fields along the flow generated by X; in particular, when applied to the metric tensor g, it provides a natural metric-compatibility criterion. The classes most relevant in this chapter are Killing, 2-Killing, and conformal vector fields, each determined by a metric-compatibility condition of Lie-derivative type [11,20]. In the remainder of this section, we define the principal classes of vector fields used in this chapter.

1. Killing vector fields. A vector field $X \in \mathfrak{X}(B)$ is Killing when it satisfies $\mathcal{L}_X g = 0$, meaning that the metric is preserved along the flow of X; therefore Killing fields generate local isometries and play a central role in geometric symmetry reduction and conservation laws [20,22].
2. 2-Killing vector fields. A vector field $X \in \mathfrak{X}(B)$ is called 2-Killing when $\mathcal{L}_X \mathcal{L}_X g = 0$, which is a second-order metric-compatibility condition extending the Killing framework and appearing in recent studies of warped-product geometry and higher-order symmetry constraints [11].
3. Conformal vector fields. A vector field $X \in \mathfrak{X}(B)$ is conformal when there exists $\phi \in C^\infty(B)$ such that $\mathcal{L}_X g = 2\phi g$; in this case the flow preserves angles while allowing local scaling, a property widely used in conformal geometry and geometric PDE [23,24].

7. Curvature and Its Principal Tensors

Curvature is the central invariant measuring non-flatness. In applications, it controls geodesic focusing, comparison inequalities, and geometric evolution equations [12,20,25]. The four objects used below are hierarchically related: Riemann tensor, sectional curvature, Ricci tensor, and scalar curvature.

7.1. Riemann Curvature Tensor

The Riemann tensor records the full local obstruction to commutation of covariant derivatives and generates the lower-order curvature invariants by contraction.

The Riemann curvature tensor is the (1,3)-tensor field

$$R: \mathfrak{X}(B) \times \mathfrak{X}(B) \times \mathfrak{X}(B) \rightarrow \mathfrak{X}(B),$$

defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{6}$$

This tensor underlies geometric PDE, Jacobi-field analysis, and curvature comparison theory [1,8].

7.2. Sectional Curvature

Sectional curvature measures curvature on individual tangent 2-planes, so it is the direct higher-dimensional analogue of Gaussian curvature. It is used in comparison geometry and rigidity theory [20,25].

For a two-plane $\sigma = \operatorname{span}\{u, v\} \subset T_p B$ with u, v linearly independent, the sectional curvature is

$$K(\sigma) = \frac{g(R(u, v)v, u)}{g(u, u)g(v, v) - g(u, v)^2} \tag{7}$$

7.3. Ricci Curvature Tensor

Ricci curvature is the average contraction of Riemann curvature and governs volume distortion along geodesic families. It plays a leading role in Ricci flow and global geometric analysis [12, 13].

The Ricci tensor is the trace of the Riemann tensor:

$$\operatorname{Ric}(Y, Z) = \operatorname{tr}(X \mapsto R(X, Y)Z). \tag{8}$$

In physics, Ricci curvature appears directly in Einstein's equations [18].

7.4. Scalar Curvature

Scalar curvature is the full metric trace of Ricci and provides a single averaged local curvature invariant. It is central in the Yamabe problem and conformal deformation theory [23, 24, 26].

The scalar curvature is the metric trace of Ricci:

$$S = \operatorname{tr}_g(\operatorname{Ric}) = g^{ij} \operatorname{Ric}_{ij} \tag{9}$$

To illustrate these constructions concretely, we compute all objects on the unit sphere. the main ideas developed in the chapter through one standard model. The unit sphere provides a concrete setting in which the manifold structure, induced metric, Levi-Civita connection, and principal curvature quantities can all be computed explicitly. In a thesis context, such a worked example helps connect abstract definitions to verifiable calculations and geometric interpretation [2,20,25]. Consider

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by $F(x, y, z) = x^2 + y^2 + z^2$. Then

$$S^2 = F^{-1}(1)$$

Furthermore,

$$\nabla F(x,y,z)=(2x,2y,2z)\neq 0 \text{ for every } (x,y,z)\in S^2.$$

Therefore 1 is a regular value of F. By the regular level-set theorem, S^2 is a smooth manifold of dimension 2. Introduce local coordinates (x_1, x_2) on the chart region $0<x_1<\pi, 0<x_2<2\pi$:

$$r(x_1,x_2)=(\sin\, x_1 \cos\, x_2, \sin\, x_1 \sin\, x_2, \cos\, x_1).$$

Differentiation gives:

$$r_{x_1}=(\cos\, x_1 \cos\, x_2, \cos\, x_1 \sin\, x_2, -\sin\, x_1), r_{x_2}=(-\sin\, x_1 \sin\, x_2, \sin\, x_1 \cos\, x_2, 0).$$

With the Euclidean inner product in $\mathbb{R}^3$:

$$g_{11}=\langle r_{x_1}, r_{x_1}\rangle=1, g_{12}=\langle r_{x_1}, r_{x_2}\rangle=0, g_{22}=\langle r_{x_2}, r_{x_2}\rangle=\sin^2\, x_1$$

Hence the induced metric is

$$g=dx_1^2+\sin^2\, x_1 dx_2^2 \tag{10}$$

The natural coordinate vector fields are ∂_{x_1} and ∂_{x_2}. Because the coefficients of (10) are independent of x_2,

$$\mathcal{L}_{\partial_{x_2}} g=0$$

Thus ∂_{x_2} is a Killing vector field and, in particular, a 2-Killing field.

From (10), the nonzero metric components and inverse components are

$$g_{11}=1, g_{22}=\sin^2\, x_1, g^{11}=1, g^{22}=\frac{1}{\sin^2\, x_1}$$

while the relevant derivatives are

$$\partial_{x_1} g_{22}=2\sin\, x_1 \cos\, x_1, \partial_{x_2} g_{11}=0, \partial_{x_2} g_{22}=0$$

Substitute these into the Levi-Civita formula

$$\Gamma_{ij}^k=\frac{1}{2} g^{k\ell}\left(\partial_i g_{j\ell}+\partial_j g_{i\ell}-\partial_\ell g_{ij}\right)$$

Then

$$\Gamma_{22}^1=\frac{1}{2} g^{11}\left(-\partial_{x_1} g_{22}\right)=-\sin\, x_1 \cos\, x_1$$

$$\Gamma_{12}^2=\frac{1}{2} g^{22} \partial_{x_1} g_{22}=\frac{1}{2} \frac{1}{\sin^2\, x_1}(2\sin\, x_1 \cos\, x_1)=\cot\, x_1$$

By the torsion-free symmetry of the Levi-Civita connection,

$$\Gamma_{21}^2=\Gamma_{12}^2$$

and the remaining Christoffel symbols vanish. Therefore,

$$\Gamma_{22}^1=-\sin\, x_1 \cos\, x_1, \Gamma_{12}^2=\Gamma_{21}^2=\cot\, x_1 \tag{11}$$

The metric tensor has matrix form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 x_1 \end{pmatrix},$$

with determinant $\det(g) = \sin^2 x_1 > 0$ for $0 < x_1 < \pi$. Hence the metric is positive-definite and (S^2, g) is a Riemannian manifold.

Using (6) together with (11), we compute

$$R^1_{212} = \partial_{x_1}\Gamma^1_{22} - \partial_{x_2}\Gamma^1_{12} + \Gamma^1_{1m}\Gamma^m_{22} - \Gamma^1_{2m}\Gamma^m_{12}.$$

Since $\Gamma^1_{12} = 0$ and $\Gamma^1_{1m} = 0$,

$R^1_{212} = \partial_{x_1}(-\sin x_1 \cos x_1) - \Gamma^1_{22}\Gamma^2_{12} = (-\cos^2 x_1 + \sin^2 x_1) - (-\sin x_1 \cos x_1)(\cot x_1) = \sin^2 x_1$.

Therefore

$$R_{1212} = g_{11} R^1_{212} = \sin^2 x_1$$

By (7),

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = \frac{\sin^2 x_1}{1 \cdot \sin^2 x_1} = 1$$

Hence the sectional curvature is constant and equal to 1 .

For the Ricci tensor, one obtains

$$\mathrm{Ric}_{11} = 1, \mathrm{Ric}_{22} = \sin^2 x_1,$$

so $\mathrm{Ric} = g$. Finally, from (9),

$$S = g^{ij}\mathrm{Ric}_{ij} = 1 + \frac{1}{\sin^2 x_1}\sin^2 x_1 = 2$$

Consequently, the round unit sphere has constant positive sectional curvature, Ricci tensor equal to the metric tensor, and scalar curvature [2,20,25].

As the introduction refers above the warped constructions provide a direct mechanism for building metrics on product spaces while retaining analytic control of connection and curvature terms. This framework is central in Riemannian geometry, Lorentzian geometry, and geometric analysis, where explicit curvature identities are required for comparison results, Einstein-type equations, and flow problems [3,4,5,25,26,27]. Frequently used families include singly warped products, doubly warped products, multiply warped products, twisted products, and sequential warped products [5,7,8,9,10,28]. The present sections focus on the singly warped, doubly warped, and sequential warped constructions.

For geometric analysis and relativity, these three types cover the standard one-step, two-factor, and two-stage warping mechanisms used in curvature computations and model construction [3,4,10,29].

Let (B_1, g_1), (B_2, g_2), and (B_3, g_3) be connected smooth Riemannian manifolds with dimensions

$$\dim B_1 = p_1, \dim B_2 = p_2, \dim B_3 = p_3.$$

For smooth positive functions $\alpha \in C^{\infty}(B_1), h \in C^{\infty}(B_2)$, and $\beta \in C^{\infty}(B_1 \times B_2)$, we use canonical lifts without changing notation.

The following notation is fixed throughout the paper:

- $\nabla, \nabla^i, \nabla^N$ denote the Levi-Civita connections on B, B_i, N, respectively.
- Rc, Rc^i, r, r_i denote Ricci and scalar curvatures.
- $H_1^{\alpha}(U_1, V_1) = g_1\left(\nabla_{U_1}^1 \mathrm{grad}^1 \alpha, V_1\right)$.
- $H_2^{h}(U_2, V_2) = g_2\left(\nabla_{U_2}^2 \mathrm{grad}^2 h, V_2\right)$.
- $H_N^{\beta}(U, V) = g_N(\nabla_U^N \mathrm{grad}^N \beta, V)$.

Throughout the paper, unless explicitly stated otherwise, B denotes the ambient manifold under consideration. Under the fixed notation of this paper, the three geometric models analyzed later are:

1. Singly warped product on $B_1 \times B_2$ with metric
$$g = g_1 \oplus \alpha^2 g_2, \alpha \in C^{\infty}(B_1), \alpha > 0.$$
2. Doubly warped product on $B_1 \times B_2$ with metric
$$g = h^2 g_1 \oplus \alpha^2 g_2, \alpha \in C^{\infty}(B_1), h \in C^{\infty}(B_2), \alpha, h > 0.$$
3. Sequential warped product on $(B_1 \times_{\alpha} B_2) \times_{\beta} B_3$ with metric
$$g = (g_1 \oplus \alpha^2 g_2) \oplus \beta^2 g_3, \beta \in C^{\infty}(B_1 \times B_2), \beta > 0$$

In each case, g is a smooth Riemannian metric and therefore determines a unique Levi-Civita connection.

8. Warped Product Manifold

Let (B_1, g_1) and (B_2, g_2) be Riemannian manifolds, and let $\alpha \in C^{\infty}(B_1)$ be positive. The warped product manifold is

$$B = B_1 \times_{\alpha} B_2$$

with metric

$$g = g_1 \oplus \alpha^2 g_2.$$

The singly warped product is a fundamental construction for generating metrics with controlled anisotropic scaling between base and fibre. It appears in radially structured geometric models, in cosmological and relativistic product-type space-times, and in geometric analysis on manifolds with separated variables [3,4,5,25]. In these settings, the Levi-Civita connection governs geodesic behaviour and covariant differentiation, the Lie derivative of the metric describes infinitesimal metric variation under flows, and the curvature tensors (Riemann, sectional, Ricci, and scalar) determine comparison properties and Einstein-type equations [9,26,27]. Accordingly, the following subsection begins with the linear connection and Levi-Civita connection formulas, which provide the basis for the subsequent Lie-derivative and curvature computations.

8.1. Linear Connection and Levi-Civita Connection

In applications of singly warped products, connection theory is essential for describing covariant differentiation, geodesic equations, transport, and curvature decomposition under warping [3,5,25]. For this reason, one first distinguishes a general linear connection from the Levi-Civita connection determined by the metric.

The difference is the following: a linear connection is any connection satisfying linearity and Leibniz rules, whereas the Levi-Civita connection is the unique linear connection that is torsion-free and metric-compatible with g[3,20,25].

For the singly warped product considered throughout this subsection, fix

$$B = B_1 \times_\alpha B_2, g = g_1 \oplus \alpha^2 g_2, \alpha \in C^\infty(B_1), \alpha > 0.$$

A linear connection D on B satisfies, for lifted fields $X_i, Y_i, Z_i \in \mathfrak{X}(B_i)$ and functions $f_i \in C^\infty(B_i) (i = 1,2,3)$, with $a, b \in \mathbb{R}$,

1. $D_{aX_i+bY_i} Z_i = aD_{X_i} Z_i + bD_{Y_i} Z_i,$
2. $D_{X_i}(aY_i + bZ_i) = aD_{X_i} Y_i + bD_{X_i} Z_i,$
3. $D_{f_i X_i} Y_i = f_i D_{X_i} Y_i,$
4. $D_{X_i}(f_i Y_i) = X_i(f_i) Y_i + f_i D_{X_i} Y_i.$

The Levi-Civita connection ∇ is characterized by the laws (for $X, Y, Z \in \mathfrak{X}(B)$)

1. $T^\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0,$
2. $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$

Under this fixed setup, for lifted fields $U_i, V_i \in \mathfrak{X}(B_i)$, the non-zero components of the Levi-Civita connection on

$$B = (B_1 \times_\alpha B_2) \times_\beta B_3$$

are given by

$$\begin{aligned}
\nabla_{U_1} V_1 &= \nabla^1_{U_1} V_1 \\
\nabla_{U_1} U_2 = \nabla_{U_2} U_1 &= U_1(\ln\ \alpha) U_2 \\
\nabla_{U_2} V_2 &= \nabla^2_{U_2} V_2 - \alpha g_2(U_2, V_2) \mathrm{grad}^1 \alpha \\
\nabla_{U_3} U_1 = \nabla_{U_1} U_3 &= U_1(\ln\ \beta) U_3 \\
\nabla_{U_3} U_2 = \nabla_{U_2} U_3 &= U_2(\ln\ \beta) U_3 \\
\nabla_{U_3} V_3 &= \nabla^3_{U_3} V_3 - \beta g_3(U_3, V_3) \mathrm{grad}^N \beta
\end{aligned}$$

These identities determine the Levi-Civita connection on sequential warped products in adapted components [3,9,28].

These fixed data for a connection identity, provide the input for Lie-derivative computations. The next subsection presents the Lie derivative of the sequential warped metric.

8.2. Lie Derivative of the Warped product manifold

The Lie derivative of the warped metric is fundamental for describing infinitesimal metric variation along vector-field flows on warped products. It separates horizontal and vertical contributions and makes explicit the role of the warping function in deformation terms. This decomposition is used in geometric analysis, including symmetry conditions and evolution equations on warped spaces. The following proposition presents the Lie Derivative of the Warped Metric.

Let $V = V_1 + V_2 \in \mathfrak{X}(B)$ and $X = X_1 + X_2, Y = Y_1 + Y_2$. Then

$$(\mathcal{L}_V g)(X, Y) = (\mathcal{L}^1_{V_1} g_1)(X_1, Y_1) + \alpha^2 (\mathcal{L}^2_{V_2} g_2)(X_2, Y_2) + 2\alpha V_1(\alpha) g_2(X_2, Y_2) \qquad (12)$$

Formula (12) is the standard decomposition for infinitesimal metric variation on warped products [3,9,10].

The study of curvature tensors on warped product manifolds is essential for understanding how warping changes intrinsic and extrinsic geometric behaviour, including comparison estimates, Einstein-type conditions, and geometric evolution equations. The following subsections present the main types of curvature tensors on the warped product manifold.

8.3. Riemann Curvature Tensor

The Riemann curvature tensor is a fundamental geometric object that measures intrinsic curvature through the non-commutativity of covariant derivatives. On warped product manifolds, it is the key input for deriving sectional, Ricci, and scalar curvature formulas and for analyzing geodesic deviation and Einstein-type conditions. We now define it mathematically as follows:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{13}$$

From the equation of Riemann Curvature Tensor above, one obtains the standard component identities

$$R(X_1,Y_1)Z_1 = R^1(X_1,Y_1)Z_1 \tag{14}$$

$$R(X_1,Y_2)Z_1 = -\frac{1}{\alpha} H_1^{\alpha}(X_1,Z_1)Y_2 \tag{15}$$

$$R(X_2,Y_2)Z_2 = R^2(X_2,Y_2)Z_2 - \|\mathrm{grad}^1 \alpha\|_1^2 \left(g_2(Y_2,Z_2)X_2 - g_2(X_2,Z_2)Y_2\right) \tag{16}$$

And the following subsection is the Sectional Curvature tensor and its important.

8.4. Sectional Curvature

Sectional curvature is important on warped product manifolds because it measures curvature along each two-dimensional direction and shows how the warping function modifies horizontal, vertical, and mixed planes. It is a central tool for geometric comparison, geodesic behavior, and curvature-sign analysis. We can define it mathematically as follows:

For a non-degenerate two-plane $\sigma = \mathrm{span}\{u,v\} \subset T_q B$,

$$K(\sigma) = \frac{g(R(u,v)v,u)}{g(u,u)g(v,v) - g(u,v)^2} \tag{17}$$

Using (14)-(16), if $u_1, v_1 \in T_x B_1$ and $u_2, v_2 \in T_y B_2$,

$$K(\mathrm{span}\{u_1,v_1\}) = K_1(\mathrm{span}\{u_1,v_1\}) \tag{18}$$

$$K(\mathrm{span}\{u_2,v_2\}) = \frac{1}{\alpha^2} K_2(\mathrm{span}\{u_2,v_2\}) - \frac{\|\mathrm{grad}^1 \alpha\|_1^2}{\alpha^2} \tag{19}$$

$$K(\mathrm{span}\{u_1,u_2\}) = -\frac{H_1^{\alpha}(u_1,u_1)}{\alpha g_1(u_1,u_1)} \tag{20}$$

These are the classical sectional-curvature formulas for singly warped products [3,5].

In the following subsection, we discuss the Ricci Curvature Tensor.

8.5. Ricci Curvature Tensor

The Ricci curvature tensor is important on warped product manifolds because it captures averaged directional curvature and directly influences volume behaviour, comparison geometry, and geometric evolution. It is used in Einstein-type equations, Ricci-flow analysis, and stability studies in geometric and physical models. It can be mathematically defined as follows:

Tracing (15)-(17) yields

$$\mathrm{Rc}(X_1,Y_1)=\mathrm{Rc}^1(X_1,Y_1)-\frac{p_2}{\alpha}H_1^{\alpha}(X_1,Y_1), \tag{21}$$

$$\mathrm{Rc}(X_2,Y_2)=\mathrm{Rc}^2(X_2,Y_2)-(\alpha\Delta_1\alpha+(p_2-1)\|\mathrm{grad}^1\alpha\|_1^2)g_2(X_2,Y_2), \tag{22}$$

$$\mathrm{Rc}(X_1,Y_2)=0. \tag{23}$$

Formulas (21)-(23) are standard and are obtained by contraction in an adapted orthonormal frame [3, 25].

8.6. Scalar Curvature

Scalar curvature is important on warped product manifolds because it summarizes total curvature at each point and provides a key invariant for global geometric analysis. Its applications include Einstein-type models, curvature comparison, and geometric evolution problems. It can be identified mathematically as follows:

The scalar curvature $r=\mathrm{tr}_g(\mathrm{Rc})$ satisfies

$$r=r_1+\frac{1}{\alpha^2}r_2-\frac{2p_2}{\alpha}\Delta_1\alpha-\frac{p_2(p_2-1)}{\alpha^2}\|\mathrm{grad}^1\alpha\|_1^2 \tag{24}$$

where r_1 and r_2 are the scalar curvatures of (B_1,g_1) and (B_2,g_2), respectively [3,5,25].

9. Doubly Warped Product Manifolds

Let (B_1,g_1) and (B_2,g_2) be Riemannian manifolds, with positive functions $\alpha\in C^{\infty}(B_1)$ and $h\in C^{\infty}(B_2)$. The doubly warped product manifold is

$$B=B_1\times_{(\alpha,h)}B_2$$

with metric

$$g=h^2g_1\oplus\alpha^2g_2$$

For the sequel, set

$$B=B_1\times_{(\alpha,h)}B_2, g=h^2g_1\oplus\alpha^2g_2,$$

with $\alpha\in C^{\infty}(B_1), h\in C^{\infty}(B_2)$, and $\alpha,h>0$. For $X=X_1+X_2$ and $Y=Y_1+Y_2$,

$$g(X,Y)=h^2g_1(X_1,Y_1)+\alpha^2g_2(X_2,Y_2), g(X_1,Y_2)=0. \tag{25}$$

Set

$$H_1^{\alpha}(U_1,V_1)=g_1\big(\nabla^1_{U_1}\mathrm{grad}^1\alpha,V_1\big), H_2^{h}(U_2,V_2)=g_2\big(\nabla^2_{U_2}\mathrm{grad}^2h,V_2\big). \tag{26}$$

Doubly warped metrics extend singly warped metrics and are treated through twisted and doubly warped constructions [3,7,8].

The doubly warped product is a natural extension of the singly warped case, where both factors are scaled and interact through two warping functions. This setting appears in geometric models with two-sided anisotropic deformation, where one needs explicit control of the Levi-Civita connection, Lie derivatives, and curvature tensors for analysis and applications [3,7,8,10]. Accordingly, the following subsection begins with the linear connection and Levi-Civita connection formulas that support the subsequent Lie-derivative and curvature computations.

9.1. Linear Connection and Levi-Civita Connection

In applications of doubly warped products, connection theory is essential for describing covariant differentiation, transport, and curvature decomposition under two warping functions [3, 7, 8]. As in the singly warped case, one distinguishes a general linear connection from the Levi-Civita connection determined by the metric.

The distinction is the same: a linear connection satisfies linearity and Leibniz rules, while the Levi-Civita connection is the unique connection that is torsion-free and metric-compatible with g[3,20,25].

For the doubly warped product considered throughout this subsection, fix

$$B = B_1 \times_{(\alpha,h)} B_2, g = h^2 g_1 \oplus \alpha^2 g_2, \alpha \in C^\infty(B_1), h \in C^\infty(B_2), \alpha, h > 0.$$

For lifted vector fields $X = X_1 + X_2$ and $Y = Y_1 + Y_2$, where $X_i, Y_i \in \mathfrak{X}(B_i)$, we use the metric split in (2.14).

A linear connection D on B satisfies, for lifted fields $X_i, Y_i, Z_i \in \mathfrak{X}(B_i)$ and functions $f_i \in C^\infty(B_i)(i = 1,2)$, with $a, b \in \mathbb{R}$,

1. $D_{aX_i + bY_i} Z_i = aD_{X_i} Z_i + bD_{Y_i} Z_i$,
2. $D_{X_i}(aY_i + bZ_i) = aD_{X_i} Y_i + bD_{X_i} Z_i$,
3. $D_{f_i X_i} Y_i = f_i D_{X_i} Y_i$,
4. $D_{X_i}(f_i Y_i) = X_i(f_i) Y_i + f_i D_{X_i} Y_i$

The Levi-Civita connection ∇ is characterized by the laws (for $X, Y, Z \in \mathfrak{X}(B)$)

1. $T^\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$,
2. $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

Under this fixed setup, for $X_i, Y_i \in \mathfrak{X}(B_i)$,

$$\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1 - g(X_1, Y_1)\operatorname{grad}(\ln h), \tag{27}$$

$$\nabla_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - g(X_2, Y_2)\operatorname{grad}(\ln \alpha), \tag{28}$$

$$\nabla_{X_1} Y_2 = \nabla_{Y_2} X_1 = X_1(\ln \alpha) Y_2 + Y_2(\ln h) X_1 \tag{29}$$

For mixed components, testing against horizontal and vertical vectors gives the two terms in (29); torsion-freeness gives symmetry in X_1, Y_2.

Equations (27)-(29) are the basic Levi-Civita identities for doubly warped products [7,8].

9.2. Lie Derivative of the Doubly Warped Metric

The Lie derivative of the doubly warped metric describes infinitesimal metric variation along flows when both warping functions are active. It separates horizontal and vertical blocks and makes explicit the deformation terms generated by α and h. The following formula gives the Lie derivative of the doubly warped metric.

For $V = V_1 + V_2$, direct differentiation of (2.14) gives

$$(\mathcal{L}_V g)(X, Y) = h^2 (\mathcal{L}^1_{V_1} g_1)(X_1, Y_1) + \alpha^2 (\mathcal{L}^2_{V_2} g_2)(X_2, Y_2) + 2hV_2(h) g_1(X_1, Y_1)$$

$$+2\alpha V_1(\alpha) g_2(X_2, Y_2) \tag{30}$$

Equation (30) is the doubly warped analogue of (1) [7,8,9].The study of curvature tensors on doubly warped product manifolds is essential for understanding the combined effect of the two warping functions on geometric behaviour. The following subsections present the main types of curvature tensors on the doubly warped product manifold.

9.3. Riemann Curvature Tensor

The Riemann curvature tensor is fundamental for quantifying intrinsic curvature under two simultaneous warping effects. We now define it mathematically as follows:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{31}$$

From the above information, together with (27)-(29), the curvature splits into factor-curvature terms plus Hessian and gradient contributions of α and h. In adapted components, one has

$$R(X_1,Y_1)Z_1 = R^1(X_1,Y_1)Z_1 - \|\mathrm{grad}^2 h\|_2^2 (g_1(Y_1,Z_1)X_1 - g_1(X_1,Z_1)Y_1), \tag{32}$$

$$R(X_2,Y_2)Z_2 = R^2(X_2,Y_2)Z_2 - \|\mathrm{grad}^1 \alpha\|_1^2 (g_2(Y_2,Z_2)X_2 - g_2(X_2,Z_2)Y_2), \tag{33}$$

$$R(X_2,X_1)Y_1 = \frac{1}{\alpha} H_1^{\alpha}(X_1,Y_1)X_2 + \frac{g_1(X_1,Y_1)}{h} \nabla_{X_2}^2 \mathrm{grad}^2 h, \tag{34}$$

$$R(X_1,X_2)Y_2 = \frac{1}{h} H_2^{h}(X_2,Y_2)X_1 + \frac{g_2(X_2,Y_2)}{\alpha} \nabla_{X_1}^1 \mathrm{grad}^1 \alpha. \tag{35}$$

Thus pure B_1 and pure B_2 components are deformed by $\mathrm{grad}^2 h$ and $\mathrm{grad}^1 \alpha$, while mixed components are controlled by H_1^{α} and H_2^{h} [7,8]. This decomposition is the starting point for sectional, Ricci, and scalar formulas below.
The following subsection is the Sectional Curvature tensor.

9.4. Sectional Curvature

Sectional curvature is important on doubly warped product manifolds because it measures curvature on two-dimensional directions and reveals how the two warping functions modify horizontal, vertical, and mixed planes. We can define it mathematically as follows:

$$K(\sigma) = \frac{g(R(u,v)v,u)}{g(u,u)g(v,v) - g(u,v)^2} \tag{36}$$

Using (32)-(35), if $u_1, v_1 \in T_x B_1$ and $u_2, v_2 \in T_y B_2$, the sectional curvature in adapted planes has the form

$$K(\mathrm{span}\{u_1,v_1\}) = \frac{1}{h^2} K_1(\mathrm{span}\{u_1,v_1\}) - \frac{\|\mathrm{grad}^2 h\|_2^2}{h^2}, \tag{37}$$

$$K(\mathrm{span}\{u_2,v_2\}) = \frac{1}{\alpha^2} K_2(\mathrm{span}\{u_2,v_2\}) - \frac{\|\mathrm{grad}^1 \alpha\|_1^2}{\alpha^2}, \tag{38}$$

and mixed two-planes involve Hessian terms of both warping functions (equivalently, second derivatives of $\ln \alpha$ and $\ln h$) [7,8].

In the following subsection, we discuss the Ricci Curvature Tensor.

9.5. Ricci Curvature Tensor

The Ricci curvature tensor is important in the doubly warped setting because it captures averaged curvature effects from both warped factors and directly enters comparison geometry and Einstein-type models. Its applications include geometric evolution equations, rigidity conditions, and stability analysis. Tracing (21)-(24) yields:

Define

$$A_\alpha = \frac{\Delta_1 \alpha}{\alpha} + (p_2 - 1)\frac{\|\mathrm{grad}^1 \alpha\|_1^2}{\alpha^2}, A_h = \frac{\Delta_2 h}{h} + (p_1 - 1)\frac{\|\mathrm{grad}^2 h\|_2^2}{h^2}. \tag{39}$$

Then the Ricci tensor in adapted components is

$$\mathrm{Rc}(X_1, Y_1) = \mathrm{Rc}^1(X_1, Y_1) - \frac{p_2}{\alpha} H_1^\alpha(X_1, Y_1) - A_h g(X_1, Y_1), \tag{40}$$

$$\mathrm{Rc}(X_2, Y_2) = \mathrm{Rc}^2(X_2, Y_2) - \frac{p_1}{h} H_2^h(X_2, Y_2) - A_\alpha g(X_2, Y_2), \tag{41}$$

$$\mathrm{Rc}(X_1, Y_2) = (p_1 + p_2 - 2) X_1(\ln\ \alpha) Y_2(\ln\ h). \tag{42}$$

The derivation follows by tracing the adapted curvature decomposition in an orthonormal frame of g[7,8].

In the following subsection, we discuss the Scalar Curvature.

9.6. Scalar Curvature

Scalar curvature is important on doubly warped product manifolds because it summarizes the total curvature contribution of both warping layers at each point. Its applications include global curvature estimates, Einstein-type equations, and geometric model comparison. The scalar curvature $r = \mathrm{tr}_g(\mathrm{Rc})$ satisfies

$$\begin{aligned} \mathrm{r} = \frac{1}{h^2} r_1 + \frac{1}{\alpha^2} r_2 - \frac{2p_2}{\alpha^2}\frac{\Delta_1 \alpha}{\alpha} - \frac{2p_1}{h^2}\frac{\Delta_2 h}{h} - \frac{p_2(p_2 - 1)}{\alpha^2}\frac{\|\mathrm{grad}^1 \alpha\|_1^2}{\alpha^2} \\ - \frac{p_1(p_1 - 1)}{h^2}\frac{\|\mathrm{grad}^2 h\|_2^2}{h^2} \end{aligned} \tag{43}$$

Thus scalar curvature is determined by the intrinsic scalar curvatures and the first and second derivatives of both warping functions [7,8].

10. Sequential Warped Product Manifolds

Let

$$N = B_1 \times_\alpha B_2, g_N = g_1 \oplus \alpha^2 g_2,$$

where $\alpha \in C^\infty(B_1)$ is positive. Let $\beta \in C^\infty(N)$ be positive. The sequential warped product manifold is

$$B = N \times_\beta B_3 = (B_1 \times_\alpha B_2) \times_\beta B_3$$

with metric

$$g = g_N \oplus \beta^2 g_3$$

Set

$$N = B_1 \times_\alpha B_2, g_N = g_1 \oplus \alpha^2 g_2$$

with $\alpha \in C^\infty(B_1)$, and let $\beta \in C^\infty(N), \beta > 0$. Define

$$B = N \times_\beta B_3, g = g_N \oplus \beta^2 g_3$$

For $X \in \mathfrak{X}(B)$, write $X = X_1 + X_2 + X_3$ with $X_i \in \mathfrak{X}(B_i)$ lifted to B. Denote by ∇^N the Levi-Civita connection of (N, g_N)[3,9,10].

The sequential warped product models two-stage geometric deformation: first on $N = B_1 \times \alpha B_2$, then on $B = N \times {}_\beta B_3$. This structure is useful in applications where curvature and flow behavior are built layer [3,9,10]. Accordingly, the following subsection begins with the linear connection and Levi-Civita connection formulas that support the subsequent Lie-derivative and curvature computations.

10.1. Linear Connection and Levi-Civita Connection

In applications of sequential warped products, connection theory determines how differentiation and transport interact with the first and second warping stages. One again distinguishes a general linear connection from the Levi-Civita connection associated with the sequential metric.

As before, a linear connection is characterized by linearity and Leibniz rules, while the Levi-Civita connection is uniquely torsion-free and metric-compatible [3,20,25].

For the sequential warped product considered throughout this subsection, fix

$$\begin{aligned} N &= B_1 \times \alpha B_2, \quad g_N = g_1 \oplus \alpha^2 g_2, \\ B &= N \times_\beta B_3, \quad g = g_N \oplus \beta^2 g_3, \alpha \in C^\infty(B_1), \beta \in C^\infty(N), \alpha, \beta > 0. \end{aligned}$$

For lifted vector fields $X = X_1 + X_2 + X_3$ and $Y = Y_1 + Y_2 + Y_3$, where $X_i, Y_i \in \mathfrak{X}(B_i)$, we use the sequential metric decomposition $g = g_N \oplus \beta^2 g_3$.

A linear connection D on B satisfies, for lifted fields $X_i, Y_i, Z_i \in \mathfrak{X}(B_i)$ and functions $f_i \in C^\infty(B_i) (i = 1,2,3)$, with $a, b \in \mathbb{R}$,

1. $D_{aX_i + bY_i} Z_i = a D_{X_i} Z_i + b D_{Y_i} Z_i$,
2. $D_{X_i}(aY_i + bZ_i) = a D_{X_i} Y_i + b D_{X_i} Z_i$,
3. $D_{f_i X_i} Y_i = f_i D_{X_i} Y_i$,
4. $D_{X_i}(f_i Y_i) = X_i(f_i) Y_i + f_i D_{X_i} Y_i$.

The Levi-Civita connection ∇ is characterized by the laws (for $X, Y, Z \in \mathfrak{X}(B)$)

1. $T^\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$,
2. $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

Under this fixed setup, for lifted fields $U_i, V_i \in \mathfrak{X}(B_i)$, the non-zero components of the Levi-Civita connection on

$$B = (B_1 \times_\alpha B_2) \times_\beta B_3$$

are given by

$$\nabla_{U_1} V_1 = \nabla^1_{U_1} V_1 \tag{44}$$

$$\nabla_{U_1} U_2 = \nabla_{U_2} U_1 = U_1(\ln\ \alpha) U_2 \tag{45}$$

$$\nabla_{U_2} V_2 = \nabla^2_{U_2} V_2 - \alpha g_2(U_2, V_2) \mathrm{grad}^1 \alpha \tag{46}$$

$$\nabla_{U_3} U_1 = \nabla_{U_1} U_3 = U_1(\ln\ \beta) U_3 \tag{47}$$

$$\nabla_{U_3} U_2 = \nabla_{U_2} U_3 = U_2(\ln\ \beta) U_3 \tag{48}$$

$$\nabla_{U_3} V_3 = \nabla^3_{U_3} V_3 - \beta g_3(U_3, V_3) \mathrm{grad}^N \beta \tag{49}$$

These identities determine the Levi-Civita connection on sequential warped products in adapted components [3,9,10].

10.2. Lie Derivative of the Doubly Warped Metric

The Lie derivative of the sequential warped metric describes infinitesimal metric variation through the two-stage warping mechanism. It separates the contributions from B_1, B_2, and B_3, and isolates the deformation terms from α and β. The following formula gives the Lie derivative of the sequential metric.

Let $V = V_1 + V_2 + V_3, X = X_1 + X_2 + X_3$, and $Y = Y_1 + Y_2 + Y_3$. Then

$$\begin{aligned}(\mathcal{L}_V g)(X,Y) = &\left(\mathcal{L}^1_{V_1} g_1\right)(X_1,Y_1) + \alpha^2\left(\mathcal{L}^2_{V_2} g_2\right)(X_2,Y_2) + 2\alpha V_1(\alpha) g_2(X_2,Y_2) \\ &+\beta^2\left(\mathcal{L}^3_{V_3} g_3\right)(X_3,Y_3) + 2\beta(V_1+V_2)(\beta) g_3(X_3,Y_3)\end{aligned}$$

The formula follows by applying the Lie-derivative decomposition first on N, then on $B = N \times {}_{\beta}B_3$[9,10].

The study of curvature tensors on sequential warped product manifolds is essential for tracking curvature transfer between the first and second warping stages. The following subsections present the main types of curvature tensors on the sequential warped product manifold.

10.3. Riemann Curvature Tensor

The Riemann curvature tensor is fundamental in the sequential setting because it records curvature interactions across the two warping levels. We now define it mathematically as follows:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{50}$$

The sequential structure yields a hierarchical decomposition as

$$R(U,W)Z = R^N(U,W)Z, U,W,Z \in \mathfrak{X}(N) \tag{51}$$

$$R(U,X_3)W = -\frac{1}{\beta} H^{\beta}_N(U,W)X_3 \tag{52}$$

$$R(X_3,Y_3)Z_3 = R^3(X_3,Y_3)Z_3 - \|\mathrm{grad}^N \beta\|^2_N (g_3(Y_3,Z_3)X_3 - g_3(X_3,Z_3)Y_3) \tag{53}$$

Here H^{β}_N is the Hessian of β on (N, g_N) [3,9,10]. Hence The following subsection is the Sectional Curvature tensor.

10.4. Sectional Curvature

Sectional curvature is important on sequential warped product manifolds because it shows how curvature is modified first on N and then on the final warped extension by β. We can define it mathematically as follows:

$$K(\sigma) = \frac{g(R(u,v)v,u)}{g(u,u)g(v,v) - g(u,v)^2} \tag{54}$$

Using (51)-(53), sectional curvature on B follows:

$$K_B(\sigma \subset TN) = K_N(\sigma) \tag{55}$$

$$K_B(\sigma \subset TB_3) = \frac{1}{\beta^2}K_3(\sigma) - \frac{\|\mathrm{grad}^N \beta\|^2_N}{\beta^2}, \tag{56}$$

$$K_B(\mathrm{span}\{u, U_3\}) = -\frac{H^{\beta}_N(u,u)}{\beta g_N(u,u)}. \tag{57}$$

Accordingly, sequential warping modifies curvature in two stages: the first through α on N, and the second through β on B[3,10].

In the following subsection, we discuss the Ricci Curvature Tensor.

10.5. Ricci Curvature Tensor

The Ricci curvature tensor is important in sequential warped products because it captures averaged curvature effects produced by both warping stages and links directly to geometric evolution and Einstein-type equations. Its applications include stability analysis, comparison geometry, and model classification. Let $U, W \in \mathfrak{X}(N)$ and $X_3, Y_3 \in \mathfrak{X}(B_3)$. Tracing (41)-(43) yields

$$\mathrm{Rc}_B(U,W) = \mathrm{Rc}_N(U,W) - \frac{p_3}{\beta} H_N^{\beta}(U,W) \tag{58}$$

$$\mathrm{Rc}_B(X_3,Y_3) = \mathrm{Rc}_3(X_3,Y_3) - \left(\beta\Delta_N\beta + (p_3-1)\|\mathrm{grad}^N\beta\|_N^2\right) g_3(X_3,Y_3) \tag{59}$$

$$\mathrm{Rc}_B(U,X_3) = 0 \tag{60}$$

Then substitute the warped formulas for Rc_N from (22)-(24) to obtain explicit B_1 and B_2 components [3,9,10].

Let $U_i, V_i \in \mathfrak{X}(B_i)$ for $i = 1,2,3$, and let H^{β} denote the Hessian of β on $N = B_1 \times_{\alpha} B_2$. In the adapted sequential decomposition, the non-zero components of Rc are

$$\mathrm{Rc}(U_1,V_1) = \mathrm{Rc}^1(U_1,V_1) - \frac{p_2}{\alpha} H_1^{\alpha}(U_1,V_1) - \frac{p_3}{\beta} H^{\beta}(U_1,V_1), \tag{61}$$

$$\mathrm{Rc}(U_2,V_2) = \mathrm{Rc}^2(U_2,V_2) - \alpha^2 g_2(U_2,V_2)\alpha^{\star} - \frac{p_3}{\beta} H^{\beta}(U_2,V_2), \tag{62}$$

$$\mathrm{Rc}(U_3,V_3) = \mathrm{Rc}^3(U_3,V_3) - \beta^2 g_3(U_3,V_3)\beta^{\star}, \tag{63}$$

$$\mathrm{Rc}\left(U_i,V_j\right) = 0, i \neq j, \tag{64}$$

where

$$\alpha^{\star} = \frac{\Delta_1\alpha}{\alpha} + (p_2-1)\frac{\|\mathrm{grad}^1\alpha\|_1^2}{\alpha^2}, \beta^{\star} = \frac{\Delta_N\beta}{\beta} + (p_1+p_2-1)\frac{\|\mathrm{grad}^N\beta\|_N^2}{\beta^2}. \tag{65}$$

These formulas are the explicit component form used in sequential warped product computations [9,10].

10.6. Scalar Curvature

Scalar curvature is an important on sequential warped product manifolds because it aggregates the full curvature effect of the first and second warping layers into a single invariant. Its applications include global geometric estimates, Einstein-type model constraints, and comparative analysis of layered warped structures. The scalar curvature $r_B = \mathrm{tr}_g(\mathrm{Rc}_B)$ satisfies

$$r_B = r_N + \frac{1}{\beta^2} r_3 - \frac{2p_3}{\beta}\Delta_N\beta - \frac{p_3(p_3-1)}{\beta^2}\|\mathrm{grad}^N\beta\|_N^2 \tag{66}$$

Using (13) on $N = B_1 \times_{\alpha} B_2$,

$$r_N = r_1 + \frac{1}{\alpha^2} r_2 - \frac{2p_2}{\alpha}\Delta_1\alpha - \frac{p_2(p_2-1)}{\alpha^2}\|\mathrm{grad}^1\alpha\|_1^2 \tag{67}$$

Substitution of (57) into (56) yields

$$
\begin{aligned}
r_B = & r_1 + \frac{1}{\alpha^2} r_2 + \frac{1}{\beta^2} r_3 - \frac{2p_2}{\alpha} \Delta_1 \alpha - \frac{p_2(p_2 - 1)}{\alpha^2} \|\mathrm{grad}^1 \alpha\|_1^2 \\
& - \frac{2p_3}{\beta} \Delta_N \beta - \frac{p_3(p_3 - 1)}{\beta^2} \|\mathrm{grad}^N \beta\|_N^2 . \qquad (68)
\end{aligned}
$$

This formula isolates the contribution of the first and second warping layers to scalar curvature [3,9,10].

11. Conclusion

This paper develops a coherent differential-geometric framework that moves from smooth manifold foundations to the metric and connection structures required for covariant differentiation, Lie derivatives, geodesic analysis, and curvature invariants. The unit-sphere computation provides a concrete validation of the formal development, confirming the Levi-Civita formulas and recovering the expected constant positive sectional curvature, Ricci tensor, and scalar curvature. Building on this base, the manuscript derives consistent identities for singly warped, doubly warped, and sequential warped products, including explicit expressions for connection components, metric Lie derivatives, and sectional, Ricci, and scalar curvatures. These results offer a unified reference for layered warped geometries and clarify how warping functions govern curvature transfer across factors. The framework is therefore suitable for further applications in geometric analysis, Einstein-type models, and relativity-oriented constructions.

ORCID iDs

Salah Agha	https://orcid.org/0009-0000-3020-4969
Ahmed k.Elsherif	https://orcid.org/0000-0003-3303-1965
Sameh Shenawy	https://orcid.org/0000-0003-3548-4239

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