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On the Geometry of Warped Product Space-Times

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ABSTRACT

The concept of smooth manifolds is a fruitful generalization of the surfaces in the Euclidean space. This concept is widely used in general relativity to generate different space-times. A space-time is pictured out as a 4-dimensional Lorentzian smooth manifold. Thus, the study of the geometry of smooth manifolds as well as space-times is of particular interest and has become a favorite topic for mathematicians and physicists. In this paper, we give some of the necessary background in differential geometry, relativity, and cosmology. This background is essential in the whole thesis. First, the concepts of differentiable as well as Riemannian manifolds are investigated. Then, we introduce the concept of warped product manifolds. Finally, as a special case of warped product manifolds, we study the geometry of generalized Robertson-Walker space-times. Einstein manifolds and their generalizations are considered. Also, Einstein-like manifolds are investigated.

1. Introduction

Let M be a differentiable manifold and T_pM be the tangent space at a point $p \in M$. Let $g_p: T_p\_M \times T_pM \to \mathbb{R}$ be an inner product for each p∈M, that is, for every $\eta_p, \vartheta_p, \varpi_p \in T_pM$ and $r \in \mathbb{R}$ the following properties hold.

1.(Symmetric) $g_p(\eta_p, \vartheta_p) = g_p(\vartheta_p, \eta_p)$;

2.(Bilinear) $g_p(\eta_p + \vartheta_p, \varpi_p) = g_p(\eta_p, \varpi_p) + g_p(\vartheta_p, \varpi_p), g_p(r\eta_p, \vartheta_p) = rg_p(\eta_p, \vartheta_p)$; and

3.(Positive-definite) $g_p(\eta_p, \eta_p) \geq 0 \; and \; g_p(\eta_p, \eta_p) = 0$ if and only if $\eta_p = 0$.

Let $\mathfrak{X}(M)$ be the tangent bundle on M whereas the set of all smooth real-valued functions on M is denoted by $\mathcal{F}(M)$. A Riemannian metric on a manifold M is a map $g: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$,

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defined by $(g(\eta,\vartheta))_p = g_p(\eta_p,\vartheta_p)$ where $g_p\left(\eta_p,\vartheta_p\right)$ is an inner product on M at p. A Riemannian manifold is a differentiable manifold M equipped with a symmetric, bilinear positive-definite metric g. If condition (3) is replaced by

(Non-degenerate) $g_p\left(\eta_p,\vartheta_p\right) = 0\ for\ all\ \vartheta_p\ implies\ \eta_p = 0,$

that is, g is symmetric, bilinear and non-degenerate, then g is called a pseudo-Riemannian metric. A pseudo-Riemannian manifold is a differentiable manifold with a pseudo-Riemannian metric g. It is clear that $g(\eta,\vartheta)$ is a smooth real-valued function on M whose value at $p \in M$ is $g_p\left(\eta_p,\vartheta_p\right)$.

A vector field η on a pseudo-Riemannian manifold M is called space-like if $g(\eta,\eta) > 0$, light-like if $g(\eta,\eta) = 0$ and time-like if $g(\eta,\eta) < 0$. It is practical to arrange the vectors on an orthonormal basis with the negative indications at the top. Let r be the number of such vector fields. The signature of a metric tensor g is $(r,n-r)$. It is noted that r is the number of negative eigenvalues of the metric tensor (g_{ij}) where $g_{ij} = g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})$ and $\{\frac{\partial}{\partial x^i}: i = 1,2,\dots,n\}$ is a basis of vector fields. For example, the signature of a Riemannian metric is $(0,n)$ and of pseudo-Riemannian is $(r,n-r)$ where $0 < r \leq n$. A Lorentzian manifold is an important special case of pseudo-Riemannian manifolds of signature $(1,n-1)$. The space-time of general relativity has a metric tensor of signature $(1,n-1)$. In local coordinates, one may write the metric tensor as follows

$$g_{ij} = g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})$$

where $\{\frac{\partial}{\partial x^i}: i = 1,2,\dots,n\}$ is a basis of vector fields. This matrix (g_{ij}) has an inverse (g^{ij}) such that

$$\delta^i_j = g_{kj}g^{ik},$$

where δ^i_j is the Kronecker delta and defined by,

$$\delta^i_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

For a pseudo-Riemannian manifold with dimension n, we have

$$\sum_{i,j=1}^{n} g_{ij}g^{ij} = n.$$

2. Covariant derivative

A linear connection ∇ on M is a function $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ denoted by $\nabla_\eta \vartheta$, such that for each $\eta, \vartheta, \varpi \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$,

1.$\nabla_\eta(\vartheta + \varpi) = \nabla_\eta \vartheta + \nabla_\eta \varpi$,

2.$\nabla_{\eta+\vartheta} \varpi = \nabla_\eta \varpi + \nabla_\vartheta \varpi$,

3.$\nabla_{f\eta} \vartheta = f \nabla_\eta \vartheta$, and

4.$\nabla_\eta(f\vartheta) = f \nabla_\eta \vartheta + (\eta f)\vartheta$.

The expression $\nabla_\eta \vartheta$ is read as the covariant derivative of ϑ in the direction of η. Now, suppose that ∇ is a linear connection on a Riemannian manifold M, then ∇ is called metric connection if the following condition holds for every $\eta, \vartheta, \varpi \in \mathfrak{X}(M)$

$$\eta g(\vartheta, \varpi) = g(\nabla_\eta \vartheta, \varpi) + g(\vartheta, \nabla_\eta \varpi),$$

where g is the metric on M. Also ∇ is said to be torsion free (or symmetric) if the torsion tensor

$$T(\eta, \vartheta) = \nabla_\eta \vartheta - \nabla_\vartheta \eta - [\eta, \vartheta],$$

vanishes for any two vector fields η and ϑ. For a smooth manifold M, a linear connection $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is said to be Riemannian (or Livi-Civita) connection if it is

1.torsion-free i.e., $\nabla_\eta \vartheta - \nabla_\vartheta \eta = [\eta, \vartheta]$, and

2.Riemannian compatible i.e., $\eta g(\vartheta, \varpi) = g(\nabla_\eta \vartheta, \varpi) + g(\vartheta, \nabla_\eta \varpi)$.

A torsion-free Riemannian compatible linear connection is unique. The covariant derivative of bases vector fields are

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k},$$

where Γ_{ij}^k is called the Christoffel symbols and it is defined as

$$\Gamma_{ij}^k = \frac{1}{2} \sum_r g^{kr} (g_{ri,j} + g_{rj,i} - g_{ij,r}),$$

where , means the partial derivative.

3. Curvature

One of the most productive ideas in differential geometry is the concept of curvature. It expands on the surfaces' Gaussian curvature. This concept is an useful geometric tool which measures the divergence of a manifold M from being Euclidean. The Riemannian curvature tensor R of a Riemannian manifold M is a correspondence that associates to every two vector fields $\eta, \vartheta \in \mathfrak{X}(M)$ a mapping

$$R(\eta,\vartheta): \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

given by [1]

$$R(\eta,\vartheta)\varpi = \nabla_\vartheta \nabla_\eta \varpi - \nabla_\eta \nabla_\vartheta \varpi + \nabla_{[\eta,\vartheta]}\varpi,$$

for each $\varpi \in \mathfrak{X}(M)$. For a Euclidean space the curvature is exactly zero, that is, $R(\eta,\vartheta)\varpi = 0$ for all vector fields η, ϑ and ϖ tangent to the Euclidean space. R is commonly recognised to have the following characteristics.

1.The Riemannian curvature R is bilinear on $\mathfrak{X}(M) \times \mathfrak{X}(M)$, that is, for $f, g \in \mathcal{F}(M)$ and $\eta_1, \eta_2, \vartheta_1, \vartheta_2 \in \mathfrak{X}(M)$, we get

$$\begin{aligned} R(f\eta_1 + g\eta_2, \vartheta_1) &= fR(\eta_1,\vartheta_1) + gR(\eta_2,\vartheta_1), \\ R(\eta_1, f\vartheta_1 + g\vartheta_2) &= fR(\eta_1,\vartheta_1) + gR(\eta_1,\vartheta_2). \end{aligned}$$

2.R is linear i.e. for $\eta, \vartheta, \varpi, T \in \mathfrak{X}(M)$ and $f, g \in \mathcal{F}(M)$, one gets

$$R(\eta,\vartheta)(f\varpi + gT) = fR(\eta,\vartheta)\varpi + gR(\eta,\vartheta)T.$$

3.For each $\eta, \vartheta, \varpi \in \mathfrak{X}(M)$, it is

$$R(\eta,\vartheta)\varpi + R(\vartheta,\varpi)\eta + R(\varpi,\eta)\vartheta = 0.$$

This identity is called the first Bianchi identity.

4.For each $\eta, \vartheta, \varpi \in \mathfrak{X}(M)$, it is

$$R(\eta,\vartheta)\varpi = -R(\vartheta,\varpi)\eta.$$

In local coordinates, the components of a Riemann curvature tensor R^m_{ijk} of a Riemannian manifold M of n-dimension are defined as

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^k} = \sum_m R^m_{ijk} \frac{\partial}{\partial x^m},$$

where

$$R^l_{ijk} = \partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ij}\Gamma^l_{km} - \Gamma^m_{ik}\Gamma^l_{jm}.$$

The Riemann curvature tensor's aforementioned characteristics can be represented in local coordinates as follows:

$$R^h_{ijk} = -R^h_{ikj},$$

$$R^k_{hij} + R^k_{ijh} + R^k_{jhi} = 0.$$

Let's introduce the useful identities of the tensor $R(\eta,\vartheta,\varpi,T) = g(R(\eta,\vartheta)\varpi,T)$. It is anti-symmetry in the first and last two positions i.e.

$$R(\eta,\vartheta,\varpi,T) = -R(\vartheta,\eta,\varpi,T) = -R(\eta,\vartheta,T,\varpi).$$

The initial Bianchi identification is valid, that is,

$$R(\eta,\vartheta,\varpi,T) + R(\vartheta,\varpi,\eta,T) + R(\varpi,\eta,\vartheta,T) = 0.$$

Finally, it is symmetric when the two vector fields are exchanged, that is,

$$R(\eta,\vartheta,\varpi,T) = R(\varpi,T,\eta,\vartheta).$$

We can define a $(0,4)$ tensor locally as

$$R_{hijk} = g_{hl}R^l_{ijk}.$$

The characteristics of this curve mentioned above have the following form. The anti-symmetries are shown as

$$R_{hijk} = -R_{ihjk}, \qquad R_{hijk} = -R_{hikj},$$

while the initial Bianchi identification is provided by

$$R_{hijk} + R_{ijhk} + R_{jhik} = 0.$$

Finally, R_{hijk} is symmetric when the two idicies are exchanged. i.e.

$$R_{hijk} = R_{jkhi}.$$

Now, it makes sense to condense the Riemann curvature tensor to produce a few new tensors. There is effectively just one contraction, called the Ricci tensor, Ric, because of the symmetries of the Riemann curvature tensor R. The Ricci tensor is defined as

$$Ric(\eta,\vartheta) = \sum_i^n g^{ii}\, g(R(\eta,e_i)\vartheta,e_i),$$

where $e_1, e_2, \dots, e_n$ be an orthonormal basis for T_pM [1]. The Ricci curvature tensor in the local coordinates is given by

$$R_{ij} = R^k_{ikj} = g^{hj}R_{hijk}.$$

The symmetries of the Riemannian curvature tensor implies that the Ricci curvature tensor is a symmetric $(0,2)$ tensor. It is

$$R_{ij} = R^k_{ikj} = g^{hj} R_{hijk} = g^{hj} R_{jkhi} = R^k_{jki} = R_{ji}.$$

A scalar curvature R is produced by a new contraction of the Ricci curvature tensor. The scalar curvature is given by

$$R = g^{ij} R_{ij}.$$

Manifolds of constant scalar curvature are those in which R is constant while manifolds with $R_{ij} = 0$ are called Ricci flat. It is evident that the scalar curvature of a Ricci flat manifold is zero.

4. Warped product manifold

Warped product manifolds are important both in differential geometry and mathematical physics, particularly in general relativity. This notion is a useful generalization of a Riemannian product manifold. Several writers have long investigated these warped product manifolds [2]. Bishop and O.Neill original introduced warped product manifolds in [3]. They made use of them to produce negative sectional curvature manifolds. Many space-times are depicted as Lorentzian warped product manifolds, with a one-dimensional base serving as the time coordinate and Riemannian fibre offering the spatial coordinates. O.Neill in [1] is the section's primary reference, unless otherwise specified. Let $(\bar{M}, \bar{g})$ and $(\tilde{M}, \tilde{g})$ be two pseudo-Riemannian manifolds with dimensions $\dim \bar{M} = \bar{n}$, $\dim \tilde{M} = \tilde{n}$ and let $f : \bar{M} \to (0; 1)$ be a smooth positive function on $\bar{M}$. Consider the product manifold $\bar{M} \times \tilde{M}$with its natural projections $\pi : \bar{M} \times \tilde{M} \to \bar{M}$ and $\eta : \bar{M} \times \tilde{M} \to \tilde{M}$. Then the warped product manifold $M = \bar{M} \times_f \tilde{M}$ is the product manifold $\bar{M} \times \tilde{M}$ furnished with metric $g = \bar{g} \oplus f^2 \tilde{g}$. The manifold $\bar{M}$ is called the base manifold of M whereas $\tilde{M}$ is called the fiber manifold of M [2, 1]. A warped product $M = \bar{M} \times_f \tilde{M}$ is called trivial if the warping function is constant. In this case, $M = \bar{M} \times_f \tilde{M}$ is the Riemannian product $M = \bar{M} \times \tilde{M}_f$, where $\tilde{M}_f$ is the manifold $\tilde{M}$ equipped with metric $f^2 \tilde{g}$, which is homothetic to $\tilde{g}$. The function $f \circ \pi$ defined on $\bar{M} \times \tilde{M}$ is called the lift of the function f to the product $\bar{M} \times \tilde{M}$. Let $p \in \bar{M}$, $q \in \tilde{M}$, and $\bar{\eta} \in T_p \bar{M}$. Then the unique vector $\eta \in T_{(p,q)}(\bar{M} \times \tilde{M})$ such that $\pi^*(\eta) = \bar{\eta}$ is called the lift of the vector field $\bar{\eta}$ to $\bar{M} \times \tilde{M}$. In this thesis, we use the same notation for a vector field and for its lift to the product manifold. Let ∇ be the Livi-Civita connection of the warped product $\bar{M} \times_f \tilde{M}$. Let ∇ and $\bar{\nabla}$ be the Livi-Civita connection of $\bar{M}$ and $\tilde{M}$ respectively. Then the Livi-Civita connection on M is given in the following result.

Proposition:-

Let $\eta, \vartheta \in \mathfrak{X}(\bar{M})$ and $V, W \in \mathfrak{X}(\tilde{M})$, then the Livi-Civita connection ∇ on $M = \bar{M} \times \tilde{M}_f$ is given by

1.$\nabla_\eta \vartheta = \bar{\nabla}_\eta \vartheta \in \mathfrak{X}(\bar{M})$,

2.$\nabla_\eta V = \nabla_V \eta = \eta(\ln f)V = V(\ln f)\eta$, and

3.$\nabla_V W = \tilde{\nabla}_V W - fg(V,W)(grad f)$.

Let $a, b, \ldots \in \{1, \ldots, \bar{n}\}$ denote the basis vector fields $\partial/_{\partial x^a}, \partial/_{\partial x^b}, \ldots$ on a neighborhood $\bar{U}$of the base manifold M whereas $\alpha, \beta, \ldots \in \{\bar{n}+1, \ldots, n\}$ denote the basis vector fields$\partial/_{\partial x^\alpha}, \partial/_{\partial x^\beta}, \ldots$on a neighborhood $\tilde{U}$ of the fiber manifold$\tilde{M}$. Likewise, $i, j, \ldots \in \{1, \ldots, n\}$ denote the basis vector fields $\partial/_{\partial x^i}, \partial/_{\partial x^j}, \ldots$ on a neighborhood $\bar{U} \times \tilde{U}$of the warped product manifold. The local components of the metric$g = \bar{g} \times_F \tilde{g}$, $F = f^2$ are $g_{ab} = \bar{g}_{ab}$, $g_{\alpha j} = 0$ and $\tilde{g}_{\alpha\beta} = F\tilde{g}_{\alpha\beta}$. The local components Γ_{ij}^h of the Livi-Civita connection on the warped product $M = \bar{M} \times_F \tilde{M}$ are as follows

$$\Gamma_{bc}^a = \bar{\Gamma}_{bc}^a, \quad \Gamma_{\beta\gamma}^\alpha = \tilde{\Gamma}_{\beta\gamma}^\alpha,$$

$$\Gamma_{\beta\gamma}^a = {}^{-1}/_2\, \bar{g}^{ab} F_b \tilde{g}_{\alpha\beta},$$

$$\Gamma_{a\beta}^\alpha = {}^1/_{2F}\, F_a \delta_\beta^\alpha,$$

$\Gamma_{ab}^\alpha = \Gamma_{\alpha b}^a = 0,$

where $F_a = \partial_a F = {}^{\partial F}/_{\partial x^a}$. Let R and Ric be the Riemannian curvature tensor and the Ricci curvature tensor of the warped product $\bar{M} \times_f \tilde{M}$, respectively and let R and Ric be the Riemann tensor and the Ricci curvature tensor of M, respectively. Likewise, let R and Ric be the Riemann tensor and the Ricci tensor of M, respectively.

The curvature of the warped product manifold $M = \bar{M} \times_f \tilde{M}$is given in terms of the curvatures on the base R and on the fibers $\tilde{R}$as follows:

Proposition: - Let $M = \bar{M} \times_f \tilde{M}$ be a pseudo-Riemannian warped product manifold. If $\eta, \vartheta, \varpi \in \mathfrak{X}(\bar{M})$ and $U, V, W \in \mathfrak{X}(\tilde{M})$, then

1.$R(\eta, \vartheta)\varpi = \bar{R}(\eta, \vartheta)\varpi \in \mathfrak{X}(\bar{M})$,

2.$R(\eta, V)\vartheta = \frac{H^f(\eta,\vartheta)}{f} V$, 3.$R(\eta,\vartheta)V = R(V,W)\eta = 0$,

4.$R(\eta,V)W = -\tilde{f}g(V,W)\bar{\nabla}_{\eta}(gradf)$, and

5.$R(V,W)U = \tilde{R}(V,W)U - \bar{g}(\bar{\nabla}f,\bar{\nabla}f)\{\tilde{g}(V,U)W - \tilde{g}(W,U)V\}$.

The local components of the Riemannian curvature tensor R_{ijkl} of the warped product $M = \bar{M} \times_F \tilde{M}$ are given by [4, 5]

$$R_{abcd} = R_{abcd},$$

$$R_{\alpha\beta\gamma\delta} = FR_{\alpha\beta\gamma\delta} - \frac{1}{4}\bar{\Delta}F\tilde{G}_{\alpha\beta\gamma\delta},$$

$$R_{\alpha ab\beta} = \frac{-1}{2} T_{ab}\tilde{g}_{\alpha\beta},$$

where $\Delta F = \bar{g}^{ab}F_aF_b$, $\tilde{G}_{\alpha\beta\gamma\delta} = \tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma}$ and T_{ab} is a $(0,2)$ tensor and its local components are $T_{ab} = \bar{\nabla}_b F_a - \frac{1}{2F}F_aF_bF$, and $T_{\alpha\beta} = T_{a\alpha} = 0$. The Ricci curvature tensor Ric of the warped product $M = \bar{M} \times_f \tilde{M}$ using the Ricci curvature tensors $\bar{R}ic$ and $\tilde{R}ic$ is considered in the following proposition.

Proposition: - Let $\eta,\vartheta \in \mathfrak{X}(\bar{M})$ and $V,W \in \mathfrak{X}(\tilde{M})$. Then the Ricci curvature of the warped product $M = \bar{M} \times_f \tilde{M}$ is

1.$Ric(\eta,\vartheta) = \bar{R}ic(\eta,\vartheta) - {}^{n}/_{f}\, H^f(\eta,\vartheta)$,

2.$Ric(\eta,V) = 0$, and

3.$Ric(V,W) = \tilde{R}ic(V,W) - f^{\circ}\tilde{g}(V,W)$, where $f^{\circ} = f\bar{\Delta}f + (\tilde{n}-1)\bar{g}(\bar{\nabla}f,\bar{\nabla}f)$.

Locally, the Ricci curvature R_{ij} of the warped product $M = \bar{M} \times_F \tilde{M}$ has the following components [4, 5]

$R_{ab} = \bar{R}_{ab} - {}^{\tilde{n}}/_{2F}\, T_{ab}$, $\quad R_{a\alpha} = 0$,

$R_{\alpha\beta} = \tilde{R}_{\alpha\beta} - (1/2)[tr(T) + \frac{\tilde{n}-1}{2F}\bar{\Delta}F]\tilde{g}_{\alpha\beta}$, where $tr(T) = \bar{g}^{ab}T_{ab}$. Finally, the scalar curvature of the warped product $M = \bar{M} \times_F \tilde{M}$ is $R = \bar{R} + ({}^{1}/_{F})\tilde{R} - ({}^{\tilde{n}}/_{F})[tr(T) + (\frac{\tilde{n}-1}{4F})\Delta F]$ [6, 5].

5. Generalized Robertson-Walker space-times

A generalized Robertson-Walker space-time, also abbreviated as GRW space-time for simplicity, is a well-known illustration of a warped product manifold. It is a generalization of the traditional Robertson-Walker space-time, abbreviated RW for short. The Lorentzian warped product $M = I \times_f \tilde{M}$ of a $(n-1)$ −Riemannian manifold $(\tilde{M},\tilde{g})$ and an open connected subinterval I of $\mathbb{R}$ with the metric $g = -dt^2 \oplus f^2\tilde{g}$, is a common representation of an n-dimensional GRW space-time, where dt^2 is the Euclidean metric on I and f is a smooth positive function. If the base manifold $\widetilde{M}$ is $n-$dimensional and has constant curvature, then the space-time becomes RW space-time. It is noted that any RW space-time is a perfect fluid space-time [1].

The Levi-Civita connection ∇ on GRW space-time may be easily derived from its metric as follows. Let $M = I \times_f \widetilde{M}$ be a generalized Robertson-Walker space-time furnished with the metric tensor $g = -dt^2 \oplus f^2\tilde{g}$. Then

$$\nabla_{\partial t}\partial t \ = \ 0,$$

$$\nabla_{\partial t}\eta \ = \ \bar{\nabla}_\eta \partial t = \frac{\dot{f}}{f}\eta,$$

$$\nabla_\eta \vartheta \ = \ \check{\nabla}_\eta \vartheta + \dot{f} f g(\eta, \vartheta)\partial t,$$

for any vector fields $\eta, \vartheta \in \mathfrak{X}(\widetilde{M})$ where ∇ is the Levi-Civita connection on M and the above dot indicates differentiation with respect to t. The Riemannian curvature tensor R of M is

$$R(\partial t, \partial t)\eta \ = \ R(\partial t, \partial t)\partial t = R(\eta, \vartheta)\partial t = 0,$$

$$R(\eta, \partial t)\partial t \ = \ \frac{\dot{f}}{f}\eta,$$

$$R(\eta, \partial t)\vartheta \ = \ \dot{f} f \tilde{g}(\eta, \vartheta)\partial t,$$

$$R(\eta, \vartheta)\varpi \ = \ \tilde{R}(\eta, \vartheta)\varpi + \dot{f}^2\{\tilde{g}(\eta, \varpi)\vartheta - \tilde{g}(\vartheta, \varpi)\eta\}.$$

The Ricci curvature tensor Ric of M is

$$Ric(\partial t, \partial t) \ = \ -\frac{\tilde{n}\ddot{f}}{f},$$

$$Ric(\eta, \partial t) \ = \ 0,$$

$$Ric(\eta, \vartheta) \ = \ \tilde{R}ic(\eta, \vartheta) - \ddot{f}\tilde{g}(\eta, \vartheta), \text{ where } f^{\circ} = f\bar{\Delta}f - (\tilde{n} - 1)\bar{g}(\bar{\nabla}f, \bar{\nabla}f).$$

Example

The Friedman-Robertson-Walker metric describes a general homogeneous and isotropic universe. In a general form1 it reads: $ds^2 = -c^2dt^2 + f^2d\sigma^2$, where

$$d\sigma^2 = \frac{1}{(1+\frac{k}{4}\eta^2)^2}(d\eta^2 + \eta^2(d\vartheta^2 + sin^2\vartheta d\phi^2)).$$

Thus, we can rewrite the metric in the following form

$$ds^2 = -c^2dt^2 + f^2\{\frac{1}{(1+\frac{k}{4}\eta^2)^2}(d\eta^2 + \eta^2(d\vartheta^2 + sin^2\vartheta d\phi^2))\},$$

with $f = f(t)$ being an arbitrary function of time only and $d\sigma^2$ being a metric of a 3-space of constant curvature. A dot denotes differentiation with respect to t, e.g. $f = \frac{df(t)}{dt}$. For simplicity, we will consider $f^2 = F$.

It is clear that $g_{tt} = -c^2, g_{ij} = \begin{bmatrix} \frac{1}{(1+\frac{k}{4}\eta^2)^2} & 0 & 0 \\ 0 & \frac{\eta^2}{(1+\frac{k}{4}\eta^2)^2} & 0 \\ 0 & 0 & \frac{\eta^2 sin^2\vartheta}{(1+\frac{k}{4}\eta^2)^2} \end{bmatrix}$ i, j = η, ϑ, ϕ.

Let us now evaluate the non-zero components of Christoffel symbol:

$$\Gamma^{\eta}_{t\eta} = \frac{1}{2F} F_t \delta^{\eta}_{\eta} = \frac{1}{2f^2} 2\dot{f}f = \frac{\dot{f}}{f}, \dots \dots \Gamma^{\vartheta}_{t\vartheta} = \Gamma^{\phi}_{t\phi} = \frac{\dot{f}}{f},$$

$$\Gamma^{t}_{\eta\eta} = \frac{-1}{2}\bar{g}^{tt} F_t \tilde{g}_{\eta\eta} = \frac{16\dot{f}f}{c^2(4+k\eta^2)^2},$$

$$\Gamma^{t}_{\vartheta\vartheta} = \frac{-1}{2}\bar{g}^{tt} F_t \tilde{g}_{\vartheta\vartheta} = \frac{16\dot{f}f\eta^2}{c^2(4+k\eta^2)^2},$$

$$\Gamma^{t}_{\phi\phi} = \frac{-1}{2}\bar{g}^{tt} F_t \tilde{g}_{\phi\phi} = \frac{16\dot{f}f\eta^2 sin^2\vartheta}{c^2(4+k\eta^2)^2},$$

$$\Gamma^{\eta}_{\eta\eta} = \tilde{\Gamma}^{\eta}_{\eta\eta} = \frac{1}{2}\tilde{g}^{\eta\eta}(\tilde{g}_{\eta\eta,\eta} + \tilde{g}_{\eta\eta,\eta} - \tilde{g}_{\eta\eta,\eta}) = \frac{-1}{2}\tilde{g}^{\eta\eta}\tilde{g}_{\eta\eta,\eta} = \frac{-2k\eta}{4+k\eta^2},$$

$$\Gamma^{\eta}_{\vartheta\vartheta} = \tilde{\Gamma}^{\eta}_{\vartheta\vartheta} = \frac{1}{2}\tilde{g}^{\eta\eta}(\tilde{g}_{\vartheta\eta,\vartheta} + \tilde{g}_{\eta\vartheta,\vartheta} - \tilde{g}_{\vartheta\vartheta,\eta}) = \frac{-1}{2}\tilde{g}^{\eta\eta}\tilde{g}_{\vartheta\vartheta,\eta} = \frac{\eta(k\eta^2-4)}{4+k\eta^2},$$

$$\Gamma^{\eta}_{\phi\phi} = \tilde{\Gamma}^{\eta}_{\phi\phi} = \frac{1}{2}\tilde{g}^{\eta\eta}(\tilde{g}_{\phi\eta,\phi} + \tilde{g}_{\eta\phi,\phi} - \tilde{g}_{\phi\phi,\eta}) = \frac{-1}{2}\tilde{g}^{\eta\eta}\tilde{g}_{\phi\phi,\eta} = \frac{\eta sin^2\vartheta(k\eta^2-4)}{4+k\eta^2},$$

$$\Gamma^{\phi}_{\eta\phi} = \frac{1}{2}\tilde{g}^{\phi\phi}(\tilde{g}_{\phi\phi,\eta} + \tilde{g}_{\eta\phi,\phi} - \tilde{g}_{\eta\phi,\phi}) = \frac{1}{2}\tilde{g}^{\phi\phi}\tilde{g}_{\phi\phi,\eta} = \frac{4-k\eta^2}{(4+k\eta^2)\eta},$$

$$\Gamma^{\vartheta}_{\eta\phi} = \frac{1}{2}\tilde{g}^{\vartheta\vartheta}(\tilde{g}_{\vartheta\vartheta,\eta} + \tilde{g}_{\eta\vartheta,\vartheta} - \tilde{g}_{\eta\vartheta,\vartheta}) = \frac{1}{2}\tilde{g}^{\vartheta\vartheta}\tilde{g}_{\vartheta\vartheta,\eta} = \frac{4-k\eta^2}{(4+k\eta^2)\eta},$$

$$\Gamma^{\phi}_{\vartheta\phi} = \frac{1}{2}\tilde{g}^{\phi\phi}(\tilde{g}_{\phi\phi,\vartheta} + \tilde{g}_{\vartheta\phi,\phi} - \tilde{g}_{\vartheta\phi,\phi}) = \frac{1}{2}\tilde{g}^{\phi\phi}\tilde{g}_{\phi\phi,\vartheta} = cot\vartheta,$$

$$\Gamma^{\vartheta}_{\phi\phi} = \frac{1}{2}\tilde{g}^{\vartheta\vartheta}(\tilde{g}_{\phi\vartheta,\phi} + \tilde{g}_{\vartheta\phi,\phi} - \tilde{g}_{\phi\phi,\vartheta}) = \frac{-1}{2}\tilde{g}^{\vartheta\vartheta}\tilde{g}_{\phi\phi,\vartheta} = -sin\vartheta cos\vartheta.$$

The non-zero components of the Riemann curvature tensor are given as:

$$R_{t\eta t\eta} = \frac{-1}{2} T_{tt}\tilde{g}_{\eta\eta} = \frac{-1}{2}\tilde{g}_{\eta\eta}[\bar{\nabla}_t F_t - \frac{1}{2F}F_t F_t] = \frac{-16\dot{f}f}{(4+k\eta^2)^2},$$

$$R_{t\eta t\eta} = \frac{-1}{2} T_{tt} \tilde{g}_{\phi\phi} = \frac{-1}{2} \tilde{g}_{\phi\phi} [\bar{\nabla}_t F_t - \frac{1}{2F} F_t F_t] = \frac{-16 \dot{f} f \eta^2 sin\vartheta}{(4+k\eta^2)^2},$$

$$R_{t\vartheta t\vartheta} = \frac{-1}{2} T_{tt} \tilde{g}_{\vartheta\vartheta} = \frac{-1}{2} \tilde{g}_{\vartheta\vartheta} [\bar{\nabla}_t F_t - \frac{1}{2F} F_t F_t] = \frac{-16 \dot{f} f \eta^2}{(4+k\eta^2)^2}.$$

Since the fiber manifold is of constant curvature, we can thus use the following relation

$$R_{\eta\phi\eta\phi} = k[\tilde{g}_{\eta\eta}\tilde{g}_{\phi\phi} - \tilde{g}_{\eta\phi}\tilde{g}_{\eta\phi}] = k\tilde{g}_{\eta\eta}\tilde{g}_{\phi\phi},$$

then

$$R_{\eta\phi\eta\phi} = F\tilde{R}_{\eta\phi\eta\phi} - \frac{1}{4}\bar{\Delta}F\tilde{G}_{\eta\phi\eta\phi} = Fk\tilde{g}_{\eta\eta}\tilde{g}_{\phi\phi} - \frac{1}{4}\left[\bar{g}_{tt}F_tF_t\tilde{g}_{\eta\eta}\tilde{g}_{\phi\phi}\right]$$

$$= 256\frac{f^2\eta^2 sin^2\vartheta(\dot{f}^2 + kc^2)}{c^2(4+k\eta^2)^4}.$$

Similarly, we can find $R_{\vartheta\phi\vartheta\phi}$and $R_{\eta\vartheta\eta\vartheta}$ by the same strategy

$$R_{\vartheta\phi\vartheta\phi} = F\tilde{R}_{\vartheta\phi\vartheta\phi} - \frac{1}{4}\bar{\Delta}F\tilde{G}_{\vartheta\phi\vartheta\phi} = Fk\tilde{g}_{\vartheta\vartheta}\tilde{g}_{\phi\phi} - \frac{1}{4}\left[\bar{g}_{tt}F_tF_t\tilde{g}_{\vartheta\vartheta}\tilde{g}_{\phi\phi}\right]$$

$$= 256\frac{f^2\eta^4 sin^2\vartheta(\dot{f}^2 + kc^2)}{c^2(4+k\eta^2)^4},$$

$$R_{\eta\vartheta\eta\vartheta} = F\tilde{R}_{\eta\vartheta\eta\vartheta} - \frac{1}{4}\bar{\Delta}F\tilde{G}_{\eta\vartheta\eta\vartheta} = Fk\tilde{g}_{\eta\eta}\tilde{g}_{\vartheta\vartheta} - \frac{1}{4}\left[\bar{g}_{tt}F_tF_t\tilde{g}_{\eta\eta}\tilde{g}_{\vartheta\vartheta}\right] = 256\frac{f^2\eta^2(\dot{f}^2 + kc^2)}{c^2(4+k\eta^2)^4}.$$

The non-zero components of the Ricci tensor are given as:

$$R_{tt} = \tilde{R}_{tt} - \frac{\tilde{n}}{2F}[\bar{\nabla}_t F_t - \frac{1}{2f^2}F_tF_t] = -3\frac{\ddot{f}}{f}.$$

$$R_{\eta\eta} = \tilde{R}_{\eta\eta} - \frac{-1}{2}[\bar{g}^{tt}T_{tt} + \frac{\tilde{n}-1}{2F}\bar{g}^{tt}F_tF_t]\tilde{g}_{\eta\eta}.$$

Since the fiber manifold is of constant curvature, thus it is Einstein. Therefore, we can use the following relation $\tilde{R}_{\eta\eta} = 2k\tilde{g}_{\eta\eta}$.

Thus,

$$R_{\eta\eta} = 2k\tilde{g}_{\eta\eta} - \frac{-1}{2}\left[\bar{g}^{tt}T_{tt} + \frac{\tilde{n}-1}{2F}\bar{g}^{tt}F_tF_t\right]\tilde{g}_{\eta\eta} = \frac{2k}{\left(1+\frac{k}{4}\eta^2\right)^2} + \frac{\ddot{f}f + 2\dot{f}^2}{c^2\left(1+\frac{k}{4}\eta^2\right)^2}$$

$$= \frac{16(\ddot{f}f + 2(\dot{f}^2 + kc^2))}{c^2(4+k\eta^2)^2}$$

Similarly, we can find $R_{\vartheta\vartheta}$ and $R_{\phi\phi}$

$$R_{\vartheta\vartheta} = \tilde{R}_{\vartheta\vartheta} - \frac{1}{2}[\bar{g}^{tt}T_{tt} + \frac{\tilde{n}-1}{2F}\bar{g}^{tt}F_tF_t]\tilde{g}_{\vartheta\vartheta} = 16\eta^2\frac{\ddot{f}f + 2(\dot{f}^2 + kc^2)}{c^2(4+k\eta^2)^2},$$

$$R_{\phi\phi} = \tilde{R}_{\phi\phi} - \frac{1}{2}[\bar{g}^{tt}T_{tt} + \frac{\tilde{n}-1}{2F}\bar{g}^{tt}F_tF_t]\tilde{g}_{\phi\phi} = 16\eta^2 sin^2\vartheta\frac{\ddot{f}f + 2(\dot{f}^2 + kc^2)}{c^2(4+k\eta^2)^2}.$$

The scalar curvature:

$$R = g^{ij}R_{ij} = g^{tt}R_{tt} + g^{\eta\eta}R_{\eta\eta} + g^{\vartheta\vartheta}R_{\vartheta\vartheta} + g^{\phi\phi}R_{\phi\phi}$$

$$= -\frac{1}{c^2}\left(-3\frac{\ddot{f}}{f}\right) + \frac{\left(1+\frac{k}{4}\eta^2\right)^2}{f^2}\left(16\frac{(\ddot{f}f + 2(\dot{f}^2+kc^2))}{c^2(4+k\eta^2)^2}\right) + \frac{\left(1+\frac{k}{4}\eta^2\right)^2}{f^2\eta^2}\left(16\eta^{2\frac{(\ddot{f}f+2(\dot{f}^2+kc^2))}{(c^2(4+k\eta^2)^2}}\right)$$

$$+ \frac{\left(1+\frac{k}{4}\eta^2\right)^2}{f^2\eta^2 sin^2\vartheta)}\left(16\eta^2 sin^2\vartheta\frac{(\ddot{f}f + 2(\dot{f}^2+kc^2))}{(c^2(4+k\eta^2)^2}\right) = \frac{3\ddot{f}}{c^2f} + 3\frac{ff + 2(f^2+kc^2)}{c^2f}$$

$$= 6\frac{ff + f^2 + kc^2}{f^2c^2}.$$

6. Standard static space-times

A standard static space-time is the Lorentzian warped product manifold $M = I_f \times \bar{M}$ equipped with the metric $g = -f^2dt^2 \oplus \bar{g}$, where f is a smooth positive function on $\bar{M}$, $(\bar{M}, \bar{g})$ is a Riemannian manifold and I is an open connected subinterval of $\mathbb{R}$ [7]. The Livi-Civita connection ∇on a standard static space-time M is given by [8]

$$\nabla_{\partial t}\partial t = f\bar{\nabla}f,$$

$$\nabla_{\partial t}\eta = \eta(ln\, f)\partial t, and$$

$$\nabla_{\eta}\vartheta = \bar{\nabla}_{\eta}\vartheta,$$

for $\eta, \vartheta \in \mathfrak{X}(\bar{M})$, where $\bar{\nabla}$ is the Livi-Civita connection on $\bar{M}$.

Now, we should know the curvatures of the standard static space-time. The Riemann curvature tensor R of M is given as follows

$$R(\partial t, \partial t)\partial t = R(\eta, \vartheta)\partial t = R(\partial t, \partial t)\eta = 0,$$

$$R(\eta, \partial t)\partial t = -f\bar{\nabla}_{\eta}gradf,$$

$$R(\partial t, \eta)\vartheta = \frac{1}{f}H^f(\eta, \vartheta)\partial t, and$$

$$R(\eta, \vartheta)\varpi = \bar{R}(\eta, \vartheta)\varpi,$$

for $\eta, \vartheta, \varpi \in \mathfrak{X}(\bar{M})$, where R is the Riemannian curvature tensor on M and $H^f(\eta, \vartheta) = \bar{g}(\bar{\nabla}_\eta grad f, \vartheta)$ is the Hessian of f.

The Ricci curvature tensor Ric of M is

$$Ric(\partial t, \partial t) = f \bar{\triangle} f,$$

$$Ric(\eta, \partial t) = 0, and$$

$$Ric(\eta, \vartheta) = \bar{R}ic(\eta, \vartheta) - \frac{1}{f} H^f(\eta, \vartheta),$$

where $\bar{\triangle} f$ is the Laplacian of the f on $\bar{M}$.

7. Einstein manifolds

The notion of warped product manifolds is a central theme in both mathematical and physical literature. It has been investigated for a long time. Curvature conditions including Einstein, locally symmetric, Ricci symmetric, and semi-symmetric warped product manifolds have received particular attention among several investigations. These circumstances have been studied in depth, and certain significant generations of Einstein manifolds are taken into consideration. A quick review of the most well-known results is taken into account in this section. The quasi-Einstein manifolds, which Chaki developed, are one of the most significant generalizations of such manifolds [9]. The quasi-Einstein manifold is defined differently by Chaki than it was previously in [10]. In general relativity literature, Lorentzian quasi-Einstein manifolds are occasionally referred to as perfect fluid manifolds. A manifold M is said to be Einstein manifold if its Ricci tensor is satisfied is well known

$$R_{ij} = \alpha g_{ij},$$

where $\alpha = \frac{R}{n}$ is a constant. In this thesis, the Einstein manifold and its generalization are thoroughly investigated. The most basic Einstein manifolds are Ricci parallel manifolds. Any generalizations of Ricci parallel manifolds are hence generalizations of Einstein manifolds. The covariant derivative of the Ricci tensor was thought to be an invariant decomposition by Gray in [11]. Seven Einstein-like metrics subspaces are produced by this decomposition. The idea of Einstein manifolds has undergone numerous generalizations that have been suggested and researched by numerous authors. Quasi-Einstein manifolds are the first generalization of the Einstein manifolds denoted by $(QE)_n$. In a $(QE)_n$ manifold M, the Ricci tensor R_{ij} has the form

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j$$

where α, β are constants and A is a non-zero 1-form [12, 9, 13], metrically equivalent to a vector field A on M, that is, $A(\eta) = g(A, \eta)$ for any vector field η. In the literature on general relativity, Lorentzian quasi-Einstein spaces are referred to as perfect fluid space-times when A^i is a unit time-like vector, that is $A_i A^i = -1$. Several mathematicians have devoted a great deal of time to studying quasi-Einstein-warped product manifolds. For instance, Dumitru examined quasi-Einstein warped product manifolds in [14]. As he studied, he acquired knowledge of the scalar curvatures R and the form of the Ricci tensor R_{ij} of the bases and fibres. Moreover, it is demonstrated that, in some circumstances, a quasi-Einstein warped product manifold reduces to a Riemannian product. The generalized quasi-Einstein manifolds are the second generalization of the Einstein manifold (denoted by $G(QE)_n$). The Ricci tensor R_{ij} of this manifold is as follows

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j,$$

where α, β, γ are constants and A, B are non-zero 1-forms [15, 16] metrically equivalent to two vector fields A, B respectively. A and B are called the generators of the manifold. Pal and Mallick in [6] investigated generalized quasi-Einstein warped product manifolds. Generalized quasi-Einstein manifolds have been defined in a wide variety of ways in the literature. For instance, in [17, 18] a generalized quasi-Einstein manifolds has a Ricci tensor of the form

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma (A_i B_j + A_j B_i),$$

where α, β, γ are constants and A, B are non-zero one forms. A mixed generalized quasi-Einstein manifold (denoted by $MG(QE)_n$) is the next generalization of the Einstein manifolds, his Ricci tensor is non-zero and satisfies

$$R_{ij} = \rho g_{ij} + \sigma A_i A_j + \lambda B_i B_j + \phi (A_i B_j + A_j B_i),$$

where $\rho, \sigma, \lambda, \phi$ are constants and A, B are two orthonormal 1-forms [19, 20, 21,22].

In terms of the warping function and the geometries of the factor manifolds, the geometry of the warped product of two pseudo-Riemannian manifolds is carefully examined (see [23, 24]). The super generalized quasi-Einstein manifolds are the next generalization of the Einstein manifolds, (denoted by $SG(QE)_n$). In these manifold the Ricci tensor is

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma (A_i B_j + A_j B_i + \delta D_{ij}),$$

where $\alpha, \beta, \gamma, \delta$ are constants, A, B are non-zero 1- forms and D_{ij} is a (0,2) symmetric tensor [25]. In [6] super generalized quasi-Einstein warped product manifold was taken into consideration. The authors gave a super generalized quasi-Einstein space-time illustration. Mixed super generalized quasi-Einstein manifolds are the last generalization of the Einstein manifold, (denoted by $MSG(QE)_n$) has a Ricci tensor of the form

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \varepsilon B_i B_j + \gamma\left(A_i B_j + A_j B_i + \delta D_{ij}\right),$$

where $\alpha, \beta, \gamma, \delta, \varepsilon$ are constants, A, B are non-zero 1-forms and D_{ij} is a $(0,2)$ symmetric tensor. B. Pal and A. Bhattacharyya investigated warped product mixed super generalized quasi-Einstein manifolds in [26]. In these manifolds, the expressions of the Ricci tensors and the scalar curvatures of the bases and fibers are found.

8. Einstein-like manifolds

In 1978, Alfard Gray shown that the gradient of the Ricci tensor $\nabla_k R_{ij}$ can be decomposed into $O(n)$ invariant terms [11] (for further details see [27, 28, 29, 30, 31]). The decomposition of the gradient of the Ricci tensor gives $O(n)$ invariant subspaces. These generated subspaces are called Gray's decomposition subspaces. Each subspace has a characteristic equation that is linear in $\nabla_k R_{ij}$. Manifolds lie in these subspaces are called Einstein-like manifolds.

Case 1 **Gray's trivial subspace**

The manifolds in the trivial subspace have parallel Ricci tensor, that is, $\nabla_k R_{ij} = 0$.

Such manifolds are called Ricci symmetric manifolds.

Case 2 Gray's subspace A

In Gray's subspace A, the Ricci tensor is Killing, that is, $\nabla_j R_{ki} + \nabla_k R_{ji} + \nabla_i R_{kj} = 0$.

A contraction of the foregoing equation is given $\nabla_k R = 0$,

this means that a manifold belongs to this subspace having constant scalar curvature.

Case 3 Gray's subspace B

In Gray's subspace B the Ricci tensor is of Codazzi type, that is, $\nabla_k R_{ij} = \nabla_i R_{kj}$,

A contraction with g^{ij} implies $\nabla_k R = 0$.

Case 4 Gray's subspace A⨁B

Gary's subspace A⨁B is distinguished by $\nabla R = 0$.

Case 5 Gray's subspace I

The gradient of the Ricci tensor in Gray's subspace I has the form

$$\nabla_k R_{ij} = \frac{n}{(n-1)(n-2)} g_{ij} \nabla_k R + \frac{n-2}{2(n-1)(n+2)} g_{jk} \nabla_i R + \frac{n-2}{(2(n-1)(n+2)} g_{ik} \nabla_j R.$$

This equation implies that the conformal curvature tensor C is divergence free. Hence, we have

$$\nabla_k R_{ij} - \nabla_j R_{ik} = \frac{1}{2(n-1)} [g_{ij} \nabla_k R - g_{ik} \nabla_j R].$$

A manifold belongs to subspace I is called Sinyukov manifold [32].

Case 6 Gray's subspace I⨁B

In Gray's subspace I⨁B, the tensor $H_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}$, is of Codazzi type. That is, the following equation holds

$$\nabla_k R_{ij} - \nabla_j R_{ik} = \frac{1}{2(n-1)} [g_{ij} \nabla_k R - g_{ik} \nabla_j R].$$

Case 7 Gray's subspace I⨁A

Gray's subspace I⨁A possesses a space-time whose Ricci tensor is conformal Killing, that is,

$$\nabla_k R_{ij} + \nabla_i R_{kj} + \nabla_j R_{ik} = \frac{2\nabla_k R}{n+2} g_{ij} + \frac{2\nabla_i R}{n+2} g_{kj} + \frac{2\nabla_j R}{n+2} g_{ik}.$$

Recently, the warped product manifold is studied in these subspaces. For example, C. A. Mantica and S. Shenawy in [30], studied Einstein-like warped product manifolds. They proved that the fiber $\tilde{M}$of the warped product manifold $M = \bar{M} \times_f \tilde{M}$ has the same Einstein-like class type of the warped product M, while the base manifold $\tilde{M}$has the same Einstein-like class type of M by imposing a sufficient condition on the warping function [30].

Again, C. A. Mantica et al [33] studied the GRW space-times in Gray's subspaces and they got the form of the Ricci tensor in each subspace. They proved that the Ricci tensor is either Einstein or takes the form of a perfect fluid space-time in all subspaces except the subspace A⨁B.

9. Doubly warped product manifold

Products that are doubly warped are generalizations of things that are single warped. In the form of warped product manifolds, Beem, Ehrilish, and Powell discovered that there exist numerous exact solutions to Einstein's field equation. Since then, singly and doubly warped product manifolds have been more crucial than ever to both physicians and mathematicians. Lorentzian doubly warped product manifolds were examined by Beem and Powell in [34]. Allison investigated the pseudo-convexity, hyperbolicity, and causal aspects of doubly warped product manifolds in [35, 36, 37]. In addition to conformally flat and conformally recurrent doubly warped product manifolds, Gebarowski also took into account doubly warped products with harmonic Weyl conformal curvature tensor in [38] and [39, 40]. The geodesic completeness of Riemannian and Lorentzian doubly warped products was studied by Bulent Unal [41]. Also, he used a doubly warped product fiber to study the hyperbolicity of generalized Robertson-Walker space-times. Unal considered various findings regarding conformal vector fields of doubly warped products in this study. Several authors in a variety of contexts, including M. Faghfouri and A. Majidi in [42], Olteanu in [43], Selcen Perktas and Erol Kilic in [44], and many more, have also investigated doubly warped product submanifolds. Lorentzian doubly warped product manifolds are a good illustration of doubly warped product space-times. These space-times are intriguing because they lead to a large number of precise solutions to Einstein's field equations.

Let (M_i, g_i, D_i) be two (pseudo-)Riemannian manifolds with metrics g_i and Levi-Civita connections D_i and let $f_i: M_i \to (0,1)$ be a positive function where $i = 1,2$. Also, suppose that $\pi_i: M_1 \times M_2 \to M_i$ is the natural projection map of the Cartesian product $M_1 \times M_2$ onto M_i where $i = 1,2$. The (pseudo-)Riemannian doubly warped product manifold $M =_{f_2} M_1 \times_{f_1} M_2$ is the product manifold $M = M_1 \times M_2$ furnished with the metric tensor $g = (f_2 \circ \pi_2)^2 \pi_1^*(g_1) \oplus (f_1 \circ \pi_1)^2 \pi_2^*(g_2)$, where $*$ denotes the pull-back operator on tensors. The functions $f_i, i = 1,2$ are called the warping functions of the warped product manifold M. In particular, if for example $f_2 = 1$, then $M = M_1 \times_{f_1} M_2$ is called a (singly) warped product manifold. A singly warped product manifold $M_1 \times_{f_1} M_2$ is said to be trivial if the warping function f_1 is also constant [44, 45, 41] and [42, 39, 46]. The Levi-Civita connection D on $M =_{f_2} M_1 \times_{f_1} M_2$ is given by

$$D_{\eta_i}\eta_j \;=\; \eta_i(ln\, f_i)\eta_j + \eta_j(ln\, f_j)\eta_i,$$

$$D_{\eta_i}\vartheta_i \;= D^i_{\eta_i}\vartheta_i \;- \frac{f_j^2}{f_i^2} g_i(\eta_i, \vartheta_i)\nabla^j(ln\, f_j),$$

where $i \neq j$ and $\eta_i, \vartheta_i \in \mathfrak{X}(M_i)$. Then the Ricci curvature tensor Ric on M is given by

$$Ric(\eta_i, \vartheta_i) = \; Ric^i(\eta_i, \vartheta_i) - \frac{n_j}{f_i} H^{f_i}(\eta_i, \vartheta_i) - \frac{f_j^\circ}{f_i^2} g_i(\eta_i, \vartheta_i),$$

$$Ric(\eta_i, \vartheta_j) \;=\; (n-2)\eta_i(ln\, f_i)\vartheta_j(ln\;\, f_j),$$

where $i \neq j$ and $\eta_i, \vartheta_i \in \mathfrak{X}(M_i)$.

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